Lecture Recording

- **Note: These lectures will be recorded and posted onto the IMPRS website**

- Dear participants,
- We will record all lectures on *"Making sense of data: introduction to statistics for gravitational wave astronomy"*, including possible Q&A after the presentation, and we will make the recordings publicly available on the IMPRS lecture website at:
  - https://imprs-gw-lectures.aei.mpg.de/2021-making-sense-of-data/
- By participating in this Zoom meeting, you are giving your explicit consent to the recording of the lecture and the publication of the recording on the course website.

Making sense of data: introduction to statistics for gravitational wave astronomy
Lecture 9: stochastic processes and sensitivity curves

AEI IMPRS Lecture Course

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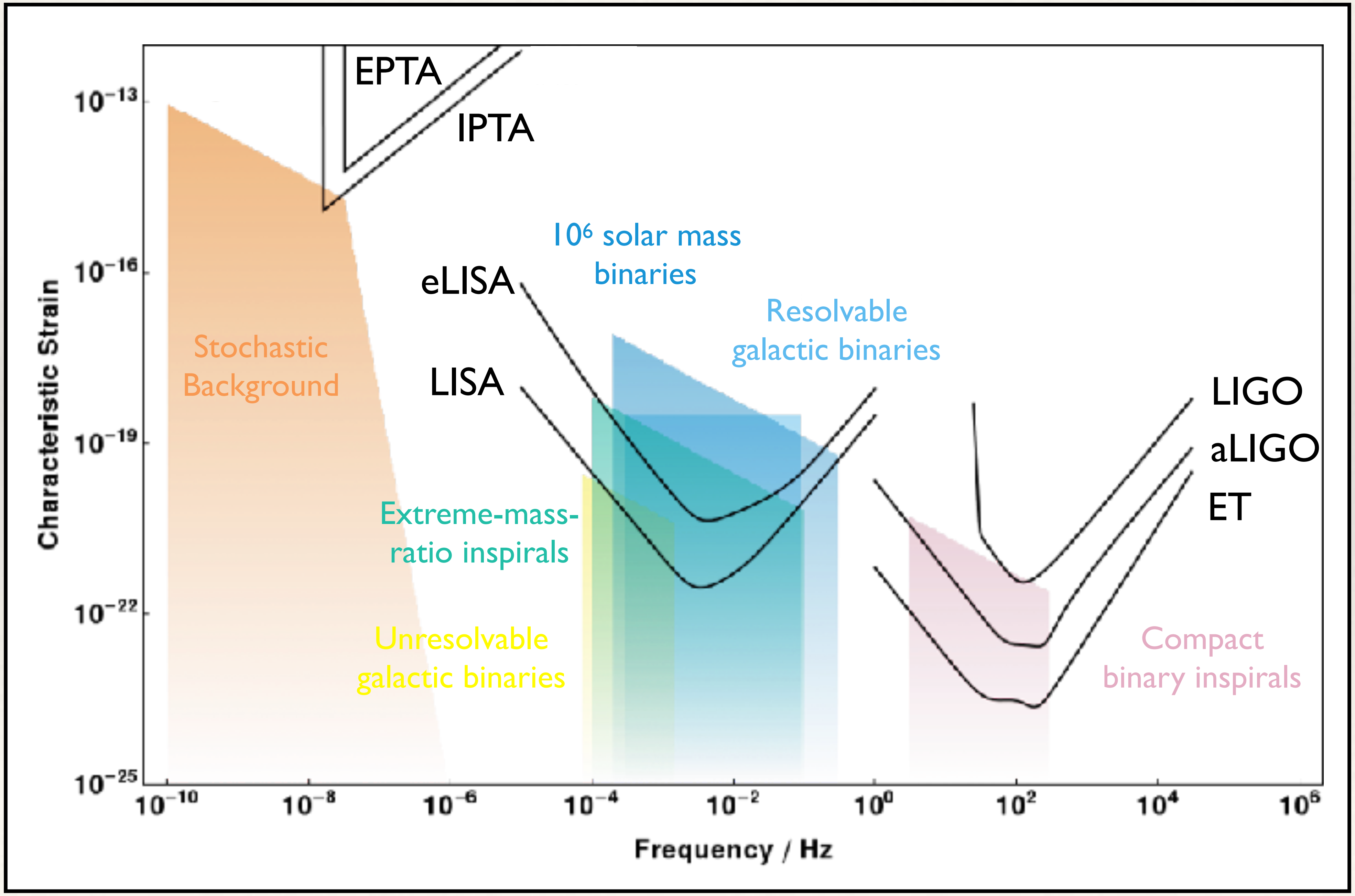


Principles of signal analysis

- Gravitational wave detectors are intrinsically noisy. The output *s(t)* will consist of a (possible) signal *h(t)* plus noise fluctuations *n(t)*.

$$s(t) = h(t) + n(t)$$

- The noise is a random process.
- Future values are not uniquely determined by initial data, but evolves according to some probabilistic model.
- We suppose the random process is drawn from an *ensemble of random processes characterised by probability distributions*

$$p_N(n_N, t_N; \ldots; n_2, t_2; n_1; t_1)\mathrm{d}n_N \ldots \mathrm{d}n_2\mathrm{d}n_1$$

Principles of signal analysis

- We typically make various useful assumptions about the properties of a random process
  - *Stationarity*: A stationary process is one for which the probability distributions depend only on time differences, not absolute time.

$$p_N(n_N, t_N + \tau; \ldots; n_2, t_2 + \tau; n_1; t_1 + \tau) = p_N(n_N, t_N; \ldots; n_2, t_2; n_1; t_1) \, \forall \tau$$

  - *Gaussianity*: A process is Gaussian if and only if all of its (absolute) probability distributions are Gaussian.

$$p_N(n_N, t_N; \ldots n_1; t_1) = A \exp\left[-\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_{jk}(n_j - \bar{n}_j)(n_k - \bar{n}_k)\right]$$

  - *Ergodicity*: An ensemble of stationary random processes is ergodic if for any process *n(t)* drawn from the ensemble, the new ensemble {*n(t+KT): K* an integer} has the same probability distributions.

Principles of signal analysis

- We are interested in how large the random fluctuations are about the mean value. We'll assume this is zero here, which can be arranged by a subtracting a constant.
- The fluctuations can be characterised by the power in a certain time interval $-T/2 < t < T/2$

$$\int_{-T/2}^{T/2} |n(t)|^2 \mathrm{d}t$$

- For stationary random processes this increases linearly with time. So, we instead use the mean power (or mean square fluctuations)

$$P_n = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |n(t)|^2 \mathrm{d}t$$

Principles of signal analysis

- Defining $n_T(t) = n(t)\mathbb{I}[|t| < T/2]$ and using Parseval's theorem we have

$$\int_{-T/2}^{T/2} [n(t)]^2 \mathrm{d}t = \int_{-\infty}^{\infty} [n_T(t)]^2 = \int_{-\infty}^{\infty} |\tilde{n}_T(f)|^2 \mathrm{d}f = 2\int_0^{\infty} |\tilde{n}_T(f)|^2 \mathrm{d}f$$

$$P_n = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [n(t)]^2 = \lim_{T\to\infty} \frac{2}{T} \int_0^{\infty} |\tilde{n}_T(f)|^2 \mathrm{d}f$$

- This motivates defining the spectral density, $S_n(f)$, via

$$S_n(f) = \lim_{T\to\infty} \frac{2}{T} \left| \int_{-T/2}^{T/2} n(t) \exp(2\pi i f t) \mathrm{d}t \right|^2$$

- This is the **one-sided spectral density** which assumes the time series is real and we only consider positive frequencies. The **two-sided spectral density** is half this.

Principles of signal analysis

- The spectral density represents the power in the process at a particular frequency

$$P_n = \int_0^\infty S_n(f)\mathrm{d}f$$

- If we consider the evolution of the process over a time interval Δt, with corresponding **bandwidth** $\Delta f = 1/\Delta t$, the mean square fluctuations in *n* at that frequency are

$$[\Delta n(\Delta t, f)]^2 \equiv \lim_{N\to\infty} \frac{2}{N} \sum_{n=-N/2}^{N/2} \left| \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} n(t) \exp(2\pi i f t)\mathrm{d}t \right|^2 = \frac{S_n(f)}{\Delta t} = S_n(f)\Delta f$$

- The *root mean square fluctuations at frequency f and measured over a time* Δt are just

$$\Delta n(\Delta t, f)_{\mathrm{rms}} = \sqrt{S_n(f)\Delta f}$$

Principles of signal analysis

- The auto-correlation function of a (zero mean) time series is defined by

$$C(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} n(t) n(t+\tau) \mathrm{d}t$$

- For an ergodic (and hence stationary) random process this is equivalent to the expectation value over the ensemble

$$C(\tau) = \langle \, n(t) \, n(t+\tau) \, \rangle$$

- The auto-correlation function is the Fourier transform of the spectral density (the Wiener-Khinchin theorem).

Principles of signal analysis

- For stationary processes a consequence of the Wiener-Khinchin theorem is that

$$\langle \tilde{n}^*(f)\tilde{n}(f')\rangle = S_n(f)\delta(f - f')$$

- where ~ denotes the Fourier transform, and * denotes complex conjugation.
- Examples of spectral densities include

$$\begin{aligned} &\textit{white noise spectrum} && S_n(f) = \text{const.} \\ &\textit{flicker noise spectrum} && S_n(f) \propto 1/f \\ &\textit{random walk spectrum} && S_n(f) \propto 1/f^2 \end{aligned}$$

Principles of signal analysis

- Can also define a **cross-spectral density** between two separate random process *n(t)* and *m(t)*

$$S_{nm}(f) = \lim_{T\to\infty} \frac{2}{T} \left[\int_{-T/2}^{T/2} n(t)\exp(-2\pi i f t)\mathrm{d}t\right] \left[\int_{-T/2}^{T/2} m(t)\exp(2\pi i f t')\mathrm{d}t'\right]$$

- Similarly we can define the cross-correlation between two time series

$$C_{nm}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} n(t)m(t+\tau)\mathrm{d}t$$

- As in the case of a single process, these are related to each other via a Fourier transform.

Principles of signal analysis

- For a Gaussian, stationary random process the spectral density conveys all the information about the statistical properties of the process.
- For gravitational wave detectors, it is natural therefore to plot the spectral density to characterise the detector sensitivity. But - how then do we represent sources on the same diagram?
- There is no unique way to do this. Different types of source are best represented in different ways.

Signal sensitivity: Bursts

- A transient burst of gravitational waves can be characterised by its **frequency**, f, its **duration**, Δt, its **bandwidth**, Δf, and its mean square amplitude, a proxy for signal power

$$\bar{P}_h = \frac{1}{\Delta t}\int_0^{\Delta t} |h(t)|^2 \mathrm{d}t = h_c^2$$

- The square root of this defines the **characteristic amplitude** of the burst, h_c.
- The power in the noise in the same bandwidth is

$$\Delta f S_n(f)$$

Signal sensitivity: Bursts

- The square root of the ratio of the signal power to the noise power is the signal-to-noise ratio.

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)^2 = \frac{\bar{P}_h}{\Delta f S_h(f)} = \frac{h_c^2}{\Delta f S_h(f)}$$

- This is a measure of detectability. If we window and bandpass the time series, this is the ratio of the root-mean-square signal contribution to the root-mean-square noise contribution.

- For a broad-band burst with $\Delta f \sim f$, the signal-to-noise ratio is approximately

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)^2 = \frac{h_c^2}{f S_h(f)}$$

- This motivates plotting $f\,S_h(f)$ instead of the power spectral density. Height above this curve is a measure of burst detectability.

Signal sensitivity: continuous waves

- Consider now a monochromatic GW source

$$h(t) = h_0 \exp(2\pi i f_0 t)$$

- The signal power is constant over time and given by

$$P_h = \lim_{T\to\infty} \int_{-T/2}^{T/2} |h(t)|^2 \mathrm{d}t = \frac{1}{2} h_0^2$$

- However, this power is concentrated at f_0. With finite time series of length T we can resolve frequency to a precision

$$\Delta f \sim 1/T$$

- Noise power in this bandwidth is $S_n(f)/T$.

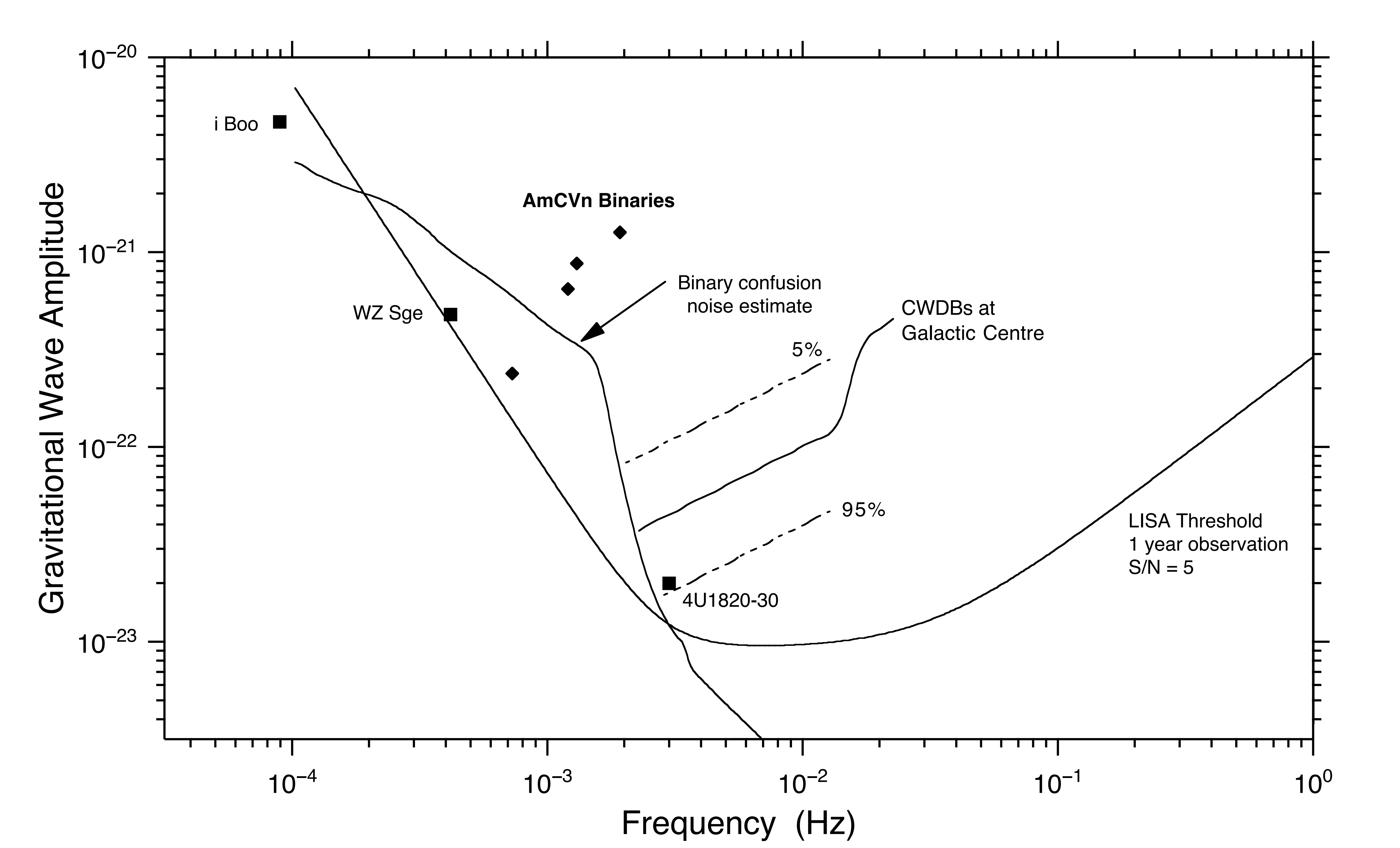
Signal sensitivity: continuous waves

- This motivates representing sensitivity by plotting

$$\sqrt{S_n(f)/T} \quad \text{or} \quad \rho_{\rm thresh}\sqrt{S_n(f)/T}$$

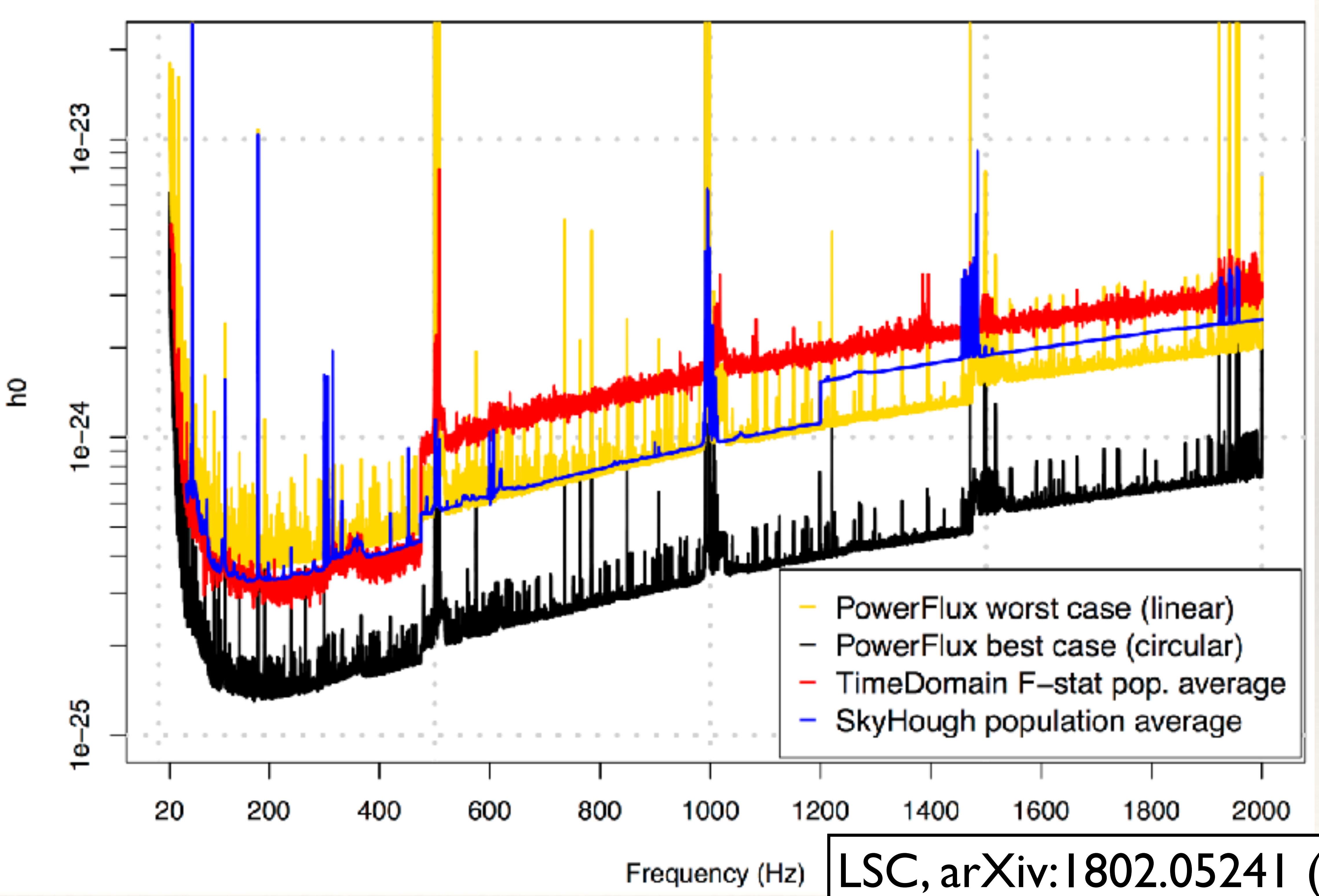
- where $\rho_{\rm thresh}$ is the estimated threshold S/N needed for detection. This is the strain spectral density.
- Advantage: for a monochromatic source, height above curve gives expected S/N or, with specified threshold, an easy assessment of whether source is detectable or not.
- Disadvantage: must specify length of observation. Not appropriate for ongoing experiments, e.g., LIGO. But can produce this after each observing run.

Signal sensitivity: continuous waves



LISA Pre-Phase A report (1998)

Signal sensitivity: continuous waves



LSC, arXiv:1802.05241 (2018)

Signal sensitivity: continuous waves

- SNRs also depend on the **sky position** and **orientation** of a source. This can be folded into the spectral density be using a *sky and orientation averaged sensitivity*, and using the strain of an optimally positioned and oriented source.

$$\langle S_h(f) \rangle_{\mathrm{SA}}^{LIGO} \approx 5 S_h(f)$$

$$\langle S_h(f) \rangle_{\mathrm{SA}}^{LISA} \approx \frac{20}{3} S_h(f)$$

Signal sensitivity: inspiraling sources

- For an inspiraling source, the total energy emitted in each frequency band is finite and so is the Fourier transform.
- Hence

$$\frac{1}{\sqrt{T}}\tilde{h}(f) \Rightarrow 0 \quad \text{as} \quad T \to \infty$$

- and so the spectral density is zero (over all time).
- Band passing and windowing can recover some of the power, but can we do better than this?
- Yes, using **filtering**.

Filtering

- A filtered time series is defined using a **kernel** *K(t-t′)*.

$$w(t) = \int_{-\infty}^{\infty} K(t - t')s(t')\mathrm{d}t'$$

- We now apply a slightly modified definition of S/N. We compare the amplitude output of the filter due to the signal to the rms output of the filter due to the noise.

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)(t) = \frac{\int_{-\infty}^{\infty} K(t - t')h(t')\mathrm{d}t'}{\sqrt{\left\langle \left| \int_{-\infty}^{\infty} K(t - t')n(t')\mathrm{d}t' \right|^2 \right\rangle}}$$

- The rms output of the filter, *S+N*, is the signal amplitude to within an rms fractional error *N/S*, which is the reciprocal of the signal to noise ratio.

Optimal filter

- We can ask what choice of filter maximises the value of *S/N* at zero-lag, i.e., *t=0*.
- From the convolution theorem for Fourier transforms we have

$$\tilde{w}(f) = \tilde{K}(f)\tilde{h}(f)$$

- The expression for S/N can thus be written

$$\frac{\mathrm{S}}{\mathrm{N}} = \frac{\int \tilde{K}(f)\tilde{h}(f)\mathrm{d}f}{\sqrt{\int |\tilde{K}(f')|^2 S_h(f')\mathrm{d}f'}}$$

Optimal filter

- This motivates a natural inner product, $(\mathbf{h}_1 \,|\, \mathbf{h}_2)$, on the space of signals of the form

$$(\mathbf{h}_1|\mathbf{h}_2) = 2\int_0^\infty \frac{\tilde{\mathbf{h}}_1(f)\tilde{\mathbf{h}}_2^*(f) + \tilde{\mathbf{h}}_1^*(f)\tilde{\mathbf{h}}_2(f)}{S_h(f)}\,\mathrm{d}f$$

- in terms of which we have

$$\frac{\mathrm{S}}{\mathrm{N}} = \frac{(S_h K|h)}{\sqrt{(S_h K|S_h K)}}$$

- which is maximised by the choice

$$\tilde{K}(f) \propto \frac{\tilde{h}(f)}{S_h(f)}$$

- This is the **Weiner optimal filter**. In the frequency domain the optimal kernel is equal to the signal weighted by the spectral density of the noise.

Optimal filter

- A search using the optimal filter then amounts to taking the inner product $(\mathbf{s} \mid \mathbf{h})$ of the data stream, $\mathbf{s}$, with a **template** of the signal, $\mathbf{h}$. This is **matched filtering**.
- The signal to noise ratio of a matched filtering search is

$$\frac{\mathrm{S}}{\mathrm{N}}[\mathbf{h}] = \frac{(\mathbf{h}|\mathbf{h})}{\sqrt{\langle(\mathbf{h}|\mathbf{n})(\mathbf{h}|\mathbf{n})\rangle}} = (\mathbf{h}|\mathbf{h})^{1/2}$$

- which follows from the fact that

$$\langle(\mathbf{h}_1|\mathbf{n})(\mathbf{h}_2|\mathbf{n})\rangle = (\mathbf{h}_1|\mathbf{h}_2)$$

- For a monochromatic source, the matched filter is just a Fourier transform, so this agrees with the previous result. In that case, the signal to noise ratio increases like the square root of the observation time.

Optimal filter

- The matched filtering $(S/N)^2$ is

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)^2 = 4\int_0^\infty \frac{|\tilde{h}(f)|^2}{S_h(f)}\mathrm{d}f$$

- which can also be written as

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)^2 = 4\int_0^\infty \frac{f|\tilde{h}(f)|^2}{S_h(f)}\mathrm{d}\ln f = 4\int_0^\infty \frac{f^2|\tilde{h}(f)|^2}{f S_h(f)}\mathrm{d}\ln f$$

- These expressions aid "integration by eye" in a logarithmic plot.
- For a source which has amplitude h_0 at frequency f and corresponding frequency derivative $\dot{f}$, we have

$$\tilde{h}(f) \sim \frac{h_0}{\sqrt{\dot{f}}}$$

Characteristic Strain

- The analogy with a broad-band burst therefore motivates the definition of a characteristic strain, h_c, for inspiraling sources (e.g., Finn and Thorne 2000).

$$h_c = h_0 \sqrt{\frac{2f^2}{\mathrm{d}f/\mathrm{d}t}}$$

- The characteristic strain is a measure of the SNR accumulated while the frequency sweeps through a bandwidth equal to frequency. If we also plot the rms noise in a bandwidth equal to frequency,

$$h_n(f) \equiv \sqrt{f \langle S_h(f) \rangle_{\mathrm{SA}}} \qquad \left(\frac{\mathrm{S}}{\mathrm{N}} \right)^2_{f \to 2f} = \left[\frac{h_c(f)}{h_n(f)} \right]^2$$

- Plots of $h_c(f)$ and $h_n(f)$ allow us to see directly how the SNR of an evolving source builds up over the evolution.

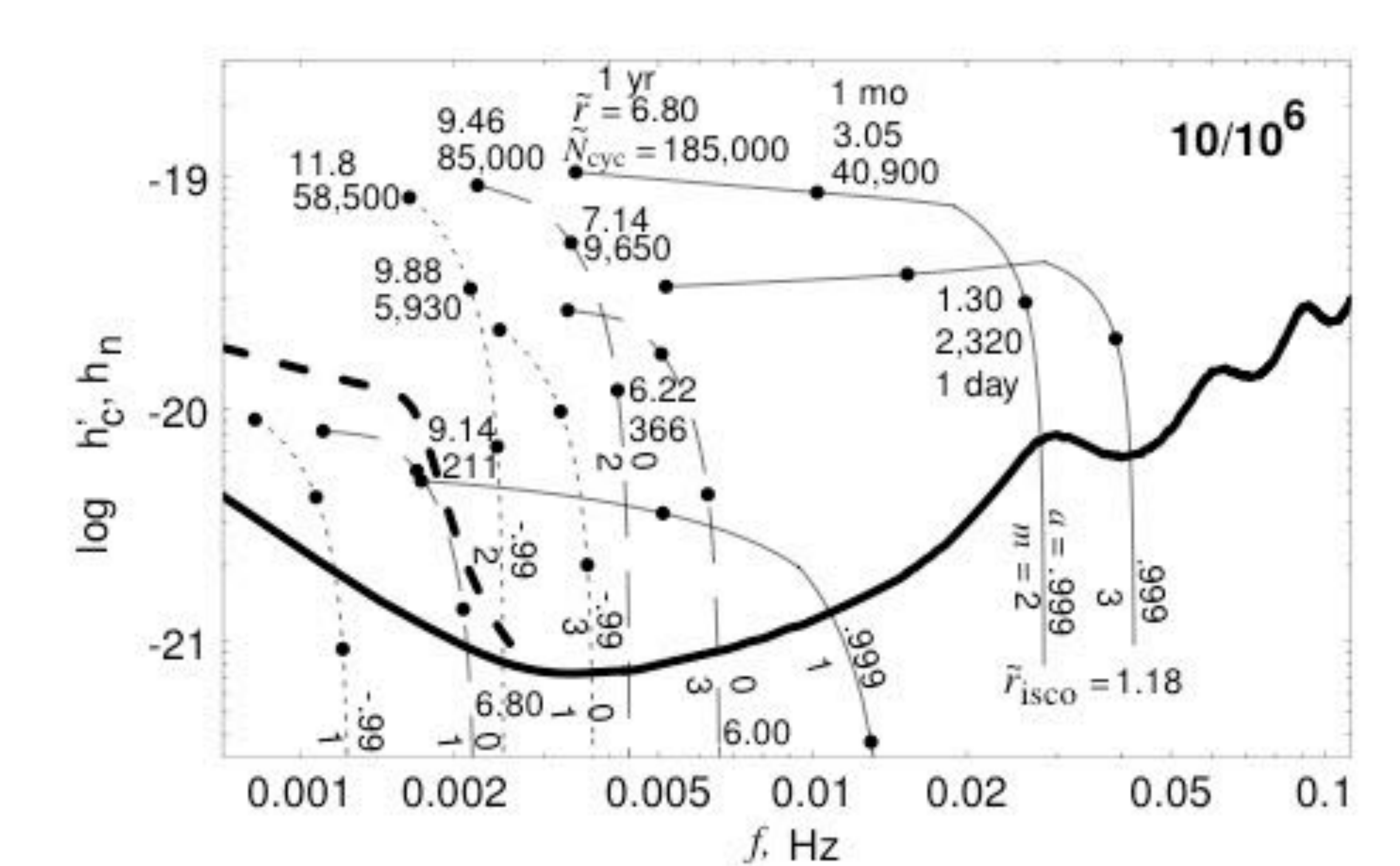
Characteristic Strain

- In the definition of characteristic strain

$$h_c = h_0 \sqrt{\frac{2 f^2}{\mathrm{d}f/\mathrm{d}t}}$$

- the term inside the square root is equal to the number of cycles the inspiral spends in the vicinity of the frequency f.
- You will read papers in which people talk about S/N being enhanced by the number of cycles spent in the vicinity of a certain frequency. This is what they mean.
- Note: plotting characteristic strain only makes sense if you are also plotting $f\, S_h(f)$. If you are plotting $S_h(f)$ directly your strain should be a factor of $\sqrt{f}$ lower.

Characteristic Strain



Build up of SNR for EMRIs observed by LISA (Finn & Thorne 2000)

Representing stochastic backgrounds

- Stochastic backgrounds are characterised by a spectral density, so it is natural to compute the power spectral density and plot it on the same axes as the detector PSD.
- There are two caveats.
    - Firstly, the “power” we have been talking about so far has not been a power in a physical sense since we have not specified any unites for the time series (and indeed for GW strain this is dimensionless). Better to use something that represents a physical energy density if possible.
    - Plotting two PSDs does not convey any information about their distinguishability. Can we represent backgrounds in a way that allows the reader to assess detectability at a glance?

Representing stochastic backgrounds

- The energy density carried by a gravitational wave is

$$\frac{\mathrm{d}E}{\mathrm{d}t\mathrm{d}A} \propto \dot{h}_+^2 + \dot{h}_\times^2$$

- Therefore, we should consider the time derivative of the strain series to get a physical energy.

- The corresponding spectral density is $f^2\, S_h(f)$ and fluctuations in a bandwidth equal to frequency are $f^3\, S_h(f)$.

- Energy densities in astrophysical and cosmological backgrounds are often expressed as a fraction of the closure density of the Universe

$$\Omega_{\mathrm{GW}} = \frac{8\pi G}{3H_0^2}\frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}\ln f} \propto f^2 h_c^2(f)$$

Representing stochastic backgrounds

- Suppose background is generated by an astrophysical population of sources with coming volume density *N(z)*. Then, total energy density in background today is

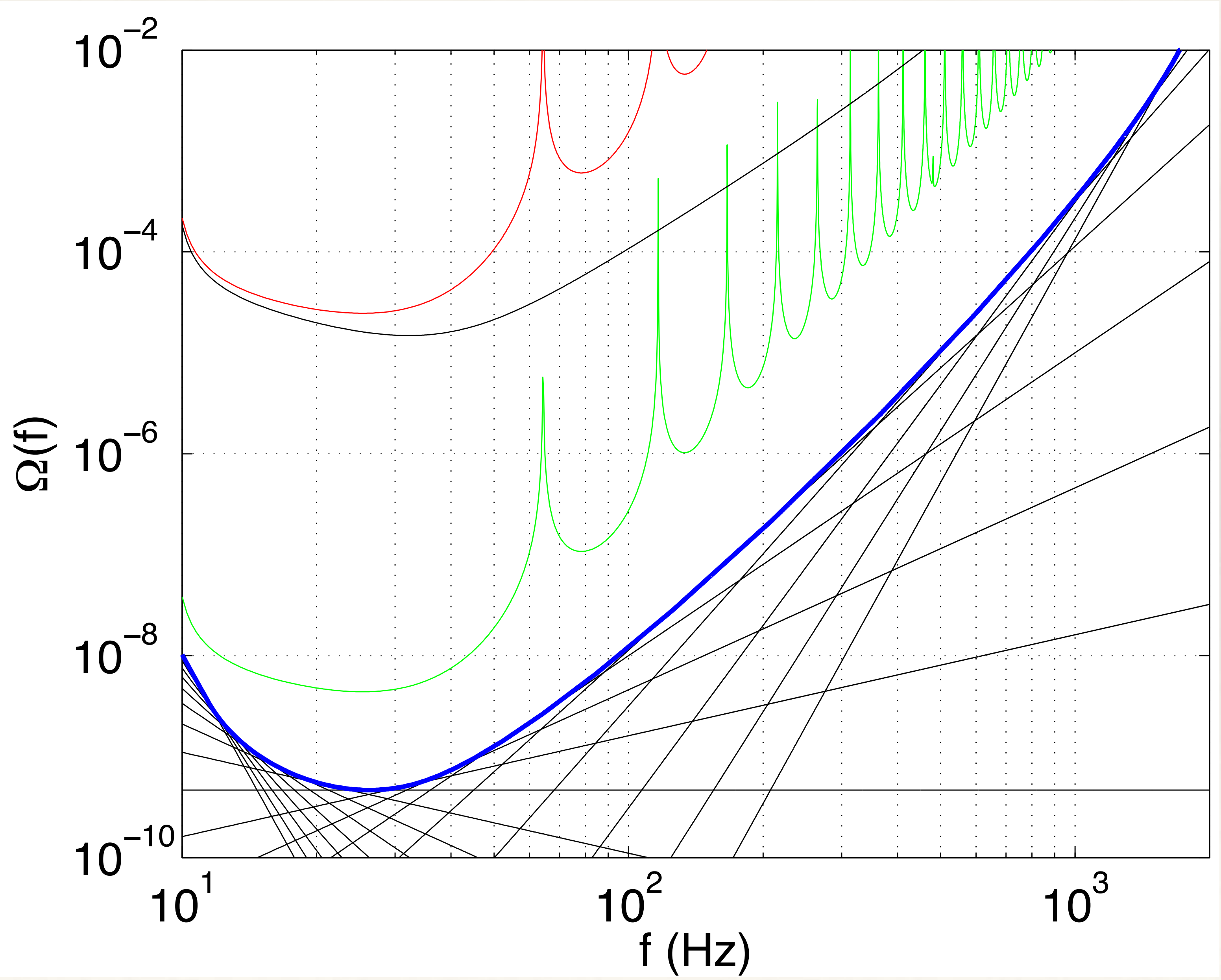
$$\mathcal{E}_{\rm GW} = \int_0^\infty \rho_c c^2 \Omega_{\rm GW} \mathrm{d}\ln f = \int_0^\infty \int_0^\infty N(z) \frac{1}{(1+z)} \frac{\mathrm{d}E}{\mathrm{d}f} f \frac{\mathrm{d}f}{f} \mathrm{d}z$$

- We deduce (Phinney 2001, astro-ph/0108028)

$$\rho_c c^2 \Omega_{\rm GW} = \frac{\pi}{4} \frac{c^2}{G} f^2 h_c^2(f) = \int_0^\infty \frac{N(z)}{1+z} \left(f_r \frac{\mathrm{d}E}{\mathrm{d}f_r} \right)_{|f_r = f(1+z)} \mathrm{d}z$$

Representing stochastic backgrounds

- Quick assessment of background detectability can be derived from power-law sensitivity curves (Thrane & Romano 2013). Requires assumptions about data analysis procedures.



Sensitivity curves: summary

- To summarise, there are four different types of sensitivity curve you might see in figures.
- **Power Spectral Density** - summarises statistical properties of noise

$$S_n(f)$$

- **Strain spectral density**

$S_n(f)/T$ - for monochromatic sources

$f S_n(f)$ - for inspirals and bursts

- **Energy spectral density** - for backgrounds

$$f^3 S_n(f)$$

Example: compact binary inspirals

- For Keplerian binaries we have

$$M = M_1 + M_2 \qquad \mu = \frac{M_1 M_2}{M_1 + M_2} \qquad r_1 M_1 = r_2 M_2 = \mu r \qquad E = -\frac{M\mu}{2r}$$

- The period is

$$\omega^2 = \left(\frac{2\pi}{T}\right)^2 = (2\pi f)^2 = \frac{M}{r^3}$$

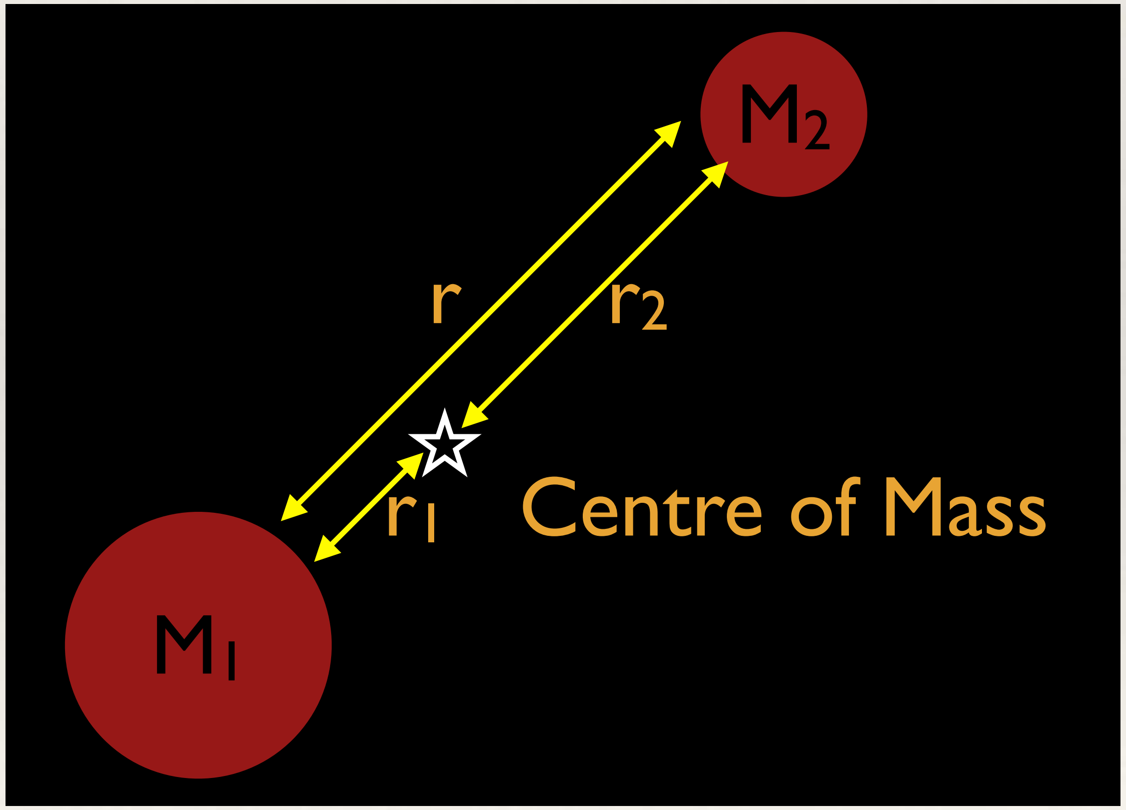
- The quadrupole moment can be estimated

$$I \sim \mu r^2 \cos 2\omega t \sim \frac{M_1 M_2}{(M_1 + M_2)^{\frac{1}{3}}} \omega^{-\frac{4}{3}}$$

- From which we deduce

$$h \sim \frac{\ddot{I}}{D} \sim \frac{1}{D} \frac{M_1 M_2}{(M_1 + M_2)^{\frac{1}{3}}} \omega^{\frac{2}{3}}$$

$$\dot{E} \sim \dddot{I}^2 \sim \mu^2 M^{\frac{4}{3}} \omega^{\frac{10}{3}}$$



Example: compact binary inspirals

- From this we obtain

$$\dot{\omega} \sim \frac{M_1 M_2}{(M_1 + M_2)^{\frac{1}{3}}} \omega^{\frac{11}{3}} = M_c^{\frac{5}{3}} \omega^{\frac{11}{3}} \qquad M_c = \frac{M_1^{\frac{3}{5}} M_2^{\frac{3}{5}}}{(M_1 + M_2)^{\frac{1}{5}}}$$

- For an individual source we have

$$\tilde{h}(f) \sim \frac{1}{D} M_c^{\frac{5}{6}} f^{-\frac{7}{6}} \qquad h_c(f) \sim \frac{1}{D} M_c^{\frac{5}{6}} f^{-\frac{1}{6}}$$

- For a background generated by inspiring binaries we have instead

$$f \frac{\mathrm{d}E}{\mathrm{d}f} \sim M_c^{\frac{5}{3}} f^{\frac{2}{3}} \qquad \Omega_{\mathrm{GW}}(f) \sim M_c^{\frac{5}{3}} f^{\frac{2}{3}} \int_0^\infty \frac{N(z)}{(1+z)^{\frac{1}{3}}} \mathrm{d}z$$

- Which yields the alternative scaling

$$h_c(f) \sim \sqrt{\Omega_{\mathrm{GW}}(f)}/f \sim M_c^{\frac{5}{6}} f^{-\frac{2}{3}} \qquad S_h(f) \sim \Omega_{\mathrm{GW}}(f)/f^3 \sim M_c^{\frac{5}{3}} f^{-\frac{7}{3}}$$