Lecture on 3+1 & BSSN formalisms of numerical relativity

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Basis equations for Numerical Relativity

$$G_{\mu\nu} = 8\pi \frac{G}{c^4} T_{\mu\nu}$$

← **Einstein's equation**

$$\begin{pmatrix} \nabla_\mu T^\mu_\nu = 0 \\ \nabla_\mu \left(\rho u^\mu\right) = 0 \\ \nabla_\mu \left(\rho u^\mu Y_l\right) = Q_l \\ \nabla_\mu F^{\mu\nu} = -4\pi j^\nu \\ \nabla_{[\mu} F_{\nu\lambda]} = 0 \\ p^\alpha \partial_\alpha f + \dot{p}^\alpha \frac{\partial f}{\partial p^\alpha} = S \end{pmatrix}$$

← **Matter-field eqs if necessary**

Greek indices: spacetime components
Latin indices: space components

Others in Numerical Relativity

- Imposing gauge conditions
- Setting initial data
- Extracting gravitational waves
- Finding black holes (finding apparent horizon)
- Adaptive mesh refinement
- A wide variety of matter-field evolution
- Messy numerics…

I will focus only on 3+1 formalism and BSSN scheme

Contents

Reference: *Numerical Relativity*, Masaru Shibata
World Scientific Publishing 2016

Section I: Introduction

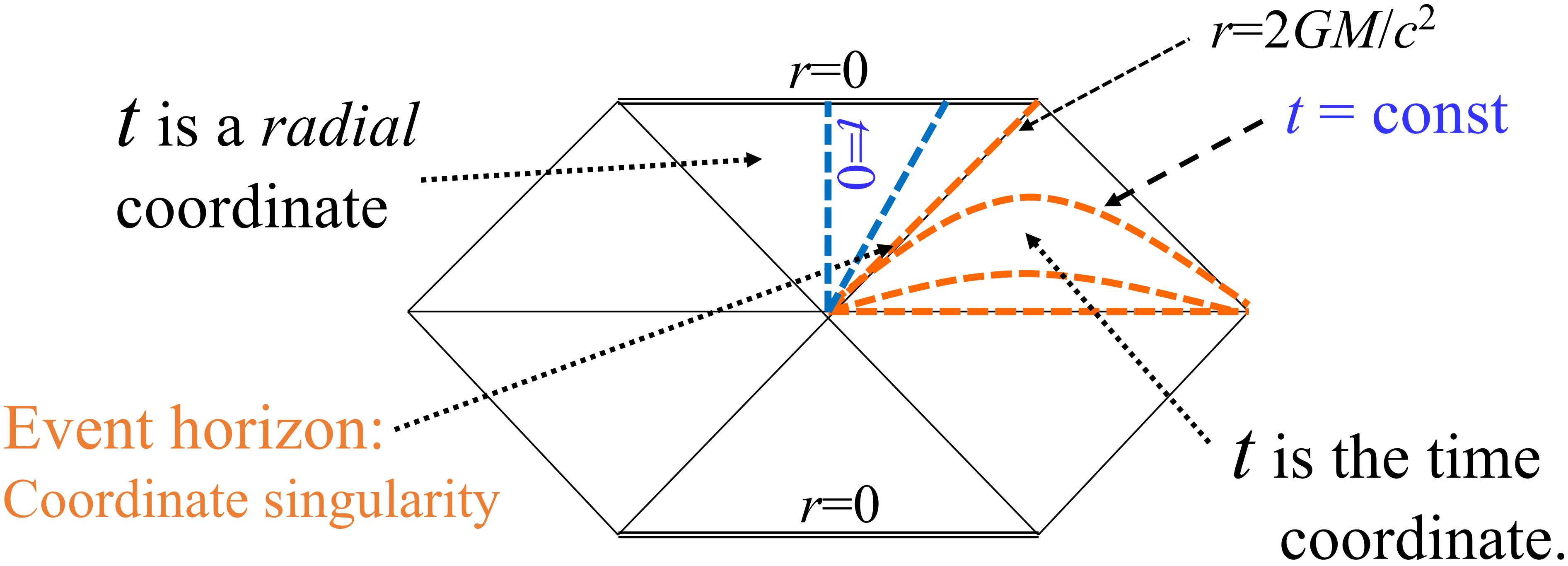
Toward the solution of Einstein's equation
for *dynamical phenomena*

- We have to solve Einstein's equations as an ***"initial value problem"***
- Einstein's equation, $G_{\mu\nu} = 8\pi G c^{-4} T_{\mu\nu}$, are *equations for space and time*
  → Space and time coordinates appear in a mixed way;
  "time coordinate" could not always have the property of time.
- E.g., for Schwarzschild coordinates;
  t is time for $r > 2Gc^{-2}M$, but is not for $r < 2Gc^{-2}M$

→ Special formalism is necessary to follow dynamics of a variety of spacetimes

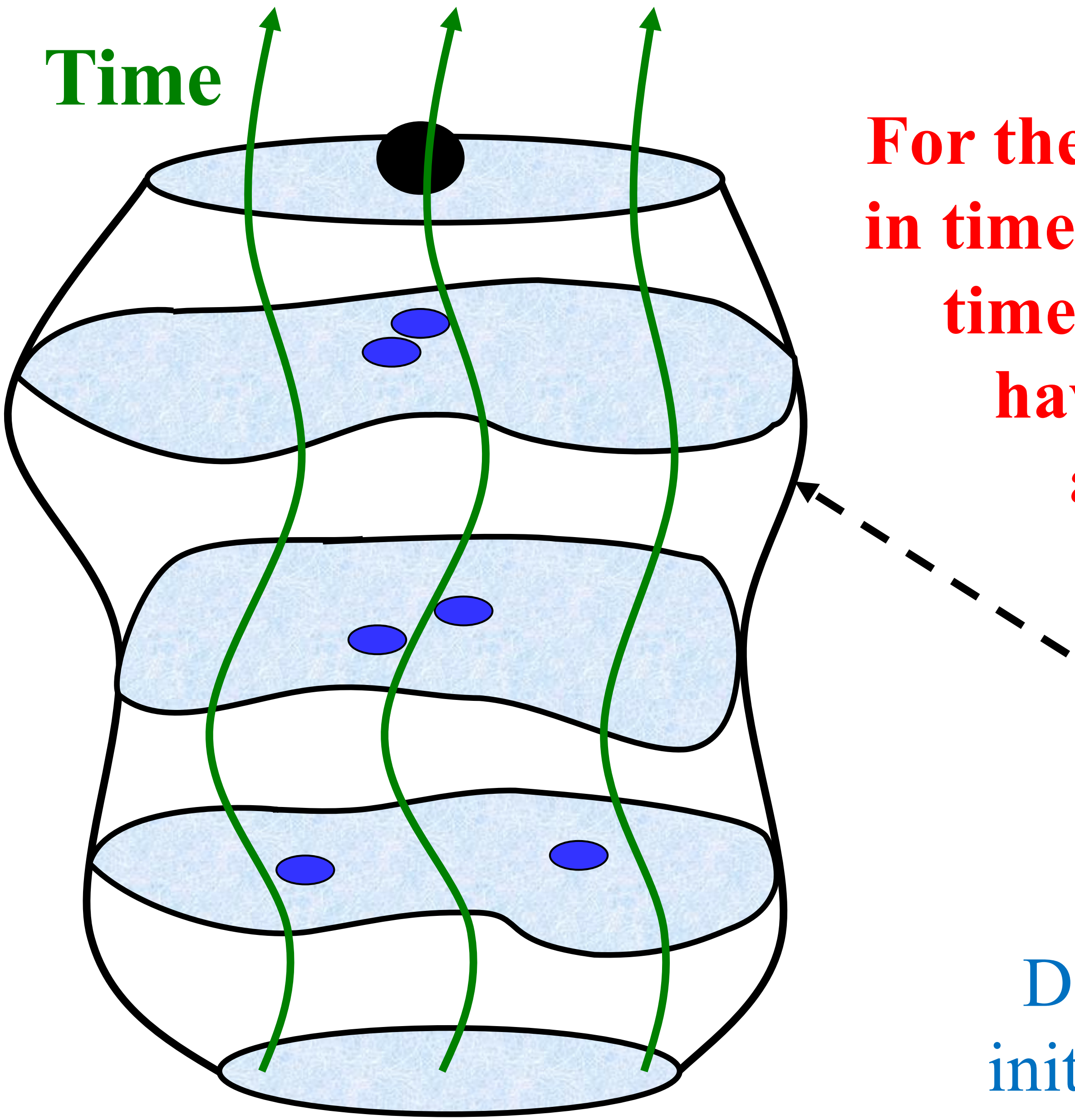
Schwarzschild spacetime

$$ds^2 = -\left(c^2 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$



Time coordinate has to have the property of "time" for dynamical evolution

For getting general dynamical spacetime



For the evolution forward in time, at each time step, time and spatial slice have to be chosen appropriately

Unknown manifold

↑

Determined by an initial-value problem

Several formulations for initial-value problems

1. **3+1 (*N*+1) formalism**
2. **Formulation based on a special (harmonic) coordinates** (often used in Post-Newtonian theory)
3. Others?

In this lecture, I will focus on the first one.

Einstein's equation = hyperbolic equations

Einstein's equation: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \frac{G}{c^4} T_{\mu\nu}$

Here, $R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}$

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)$$

$$\rightarrow 2(-g)G^{\mu\nu} = \partial_\alpha\partial_\beta\left[(-g)\left(g^{\mu\nu}g^{\alpha\beta} - g^{\alpha\mu}g^{\beta\nu}\right)\right] + (-g)t^{\mu\nu}$$

$t^{\mu\nu} = O\left\{\left(\partial g_{\mu\nu}\right)^2\right\}$: Pseudo tensor of Landau & Lifshitz

De Donder gauge: $\partial_\alpha\left(\sqrt{-g}\, g^{\alpha\beta}\right) = 0$

$$\rightarrow 2(-g)G^{\mu\nu} = \sqrt{-g}\, g^{\alpha\beta}\partial_\alpha\partial_\beta\left(\sqrt{-g}\, g^{\mu\nu}\right) + O\left\{\left(\partial g_{\mu\nu}\right)^2\right\}$$

↓

Wave equation

Einstein's equation is similar to multi-component scalar wave equation: several reformulations

① $\partial_t^2 \phi_a = \Delta \phi_a$ ⟷ **Post-Newtonian way**

② $\rightarrow \begin{cases} \eta_a = \partial_t \phi_a \\ \partial_t \eta_a = \Delta \phi_a \end{cases}$ ⟷ **3+1 way**

$(\eta_a, \phi_a) \Leftrightarrow (K_{ij}, \gamma_{ij})$

where a=1—N: # of components

or

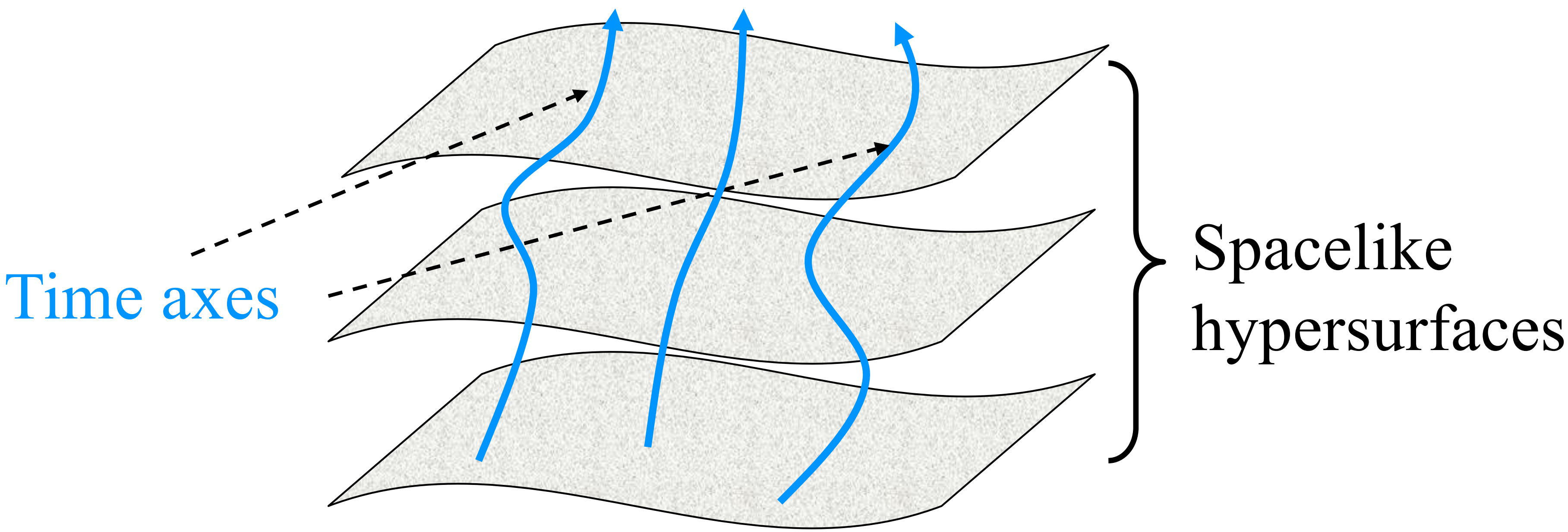
③ $\rightarrow \begin{cases} \partial_t \xi_{a\,i} = \partial_i \phi_a \\ \partial_t \phi_a = \partial_i \xi_a{}^i \end{cases}$ ⟷ **Hyperbolic way**

Similar to each other: However, spacetime is not a priori given in general relativity

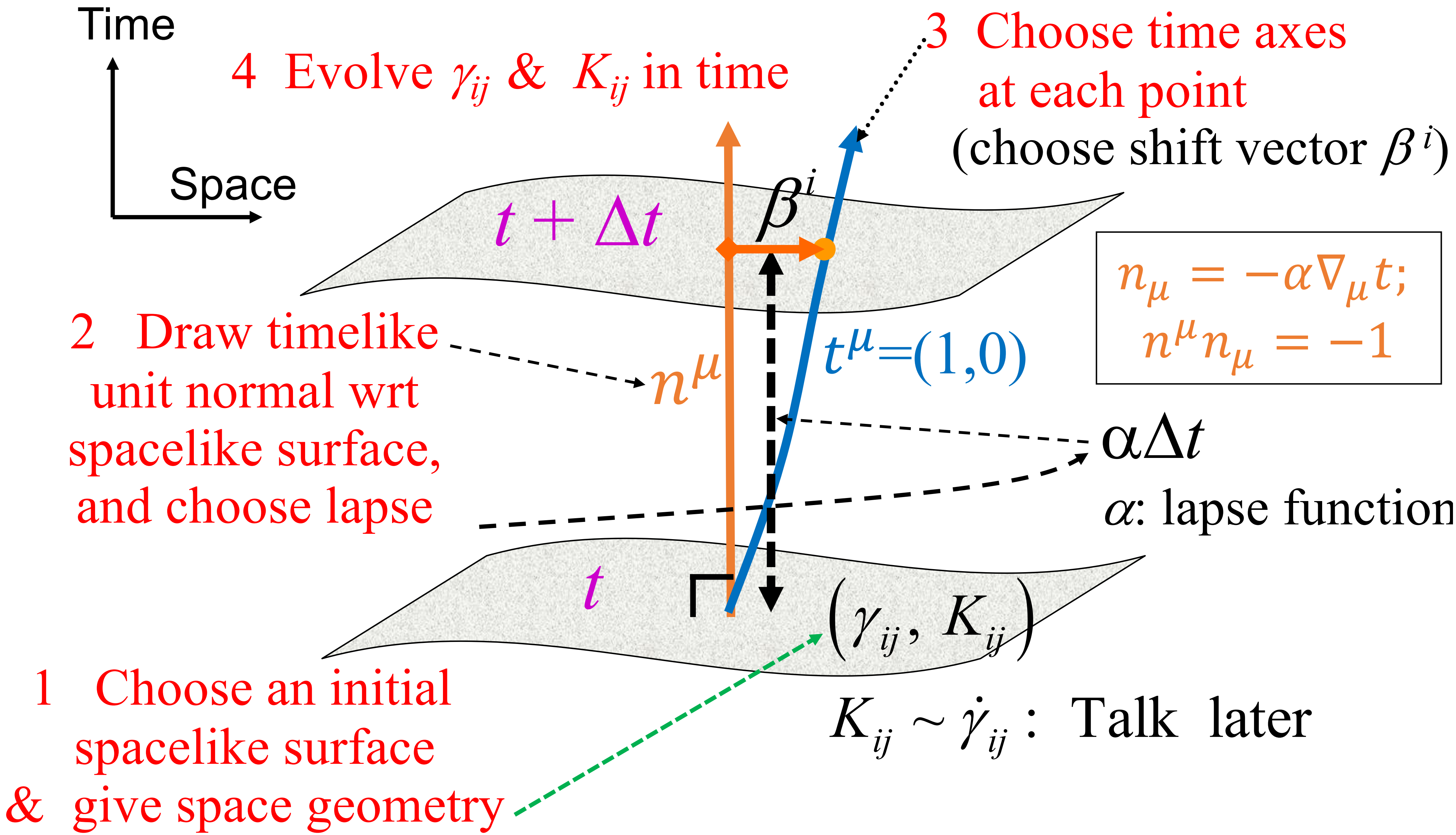
Section II: 3+1 (ADM) formalism

Concept

1. Foliate spacetime by spacelike surfaces
2. "Choose" appropriate time coordinates for a direction of time at each location on the slice
3. Follow dynamics of spacelike surfaces



First step: 3+1 decomposition & definition of variables



Note: $\alpha n^\mu = t^\mu - \beta^\mu$

Defined variables

γ_{ij} = space metric
K_{ij} = extrinsic curvature
α = lapse function
β^i = shift vector

$g_{\mu\nu} \rightarrow (\gamma_{ij}, \alpha, \beta^k)$

- Dynamics (γ_{ij}, K_{ij})
- Gauge (α, β^i)

$$\gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu: \quad \gamma_{\mu\nu} n^\nu = 0 \quad \& \; n_\mu n^\mu = -1$$

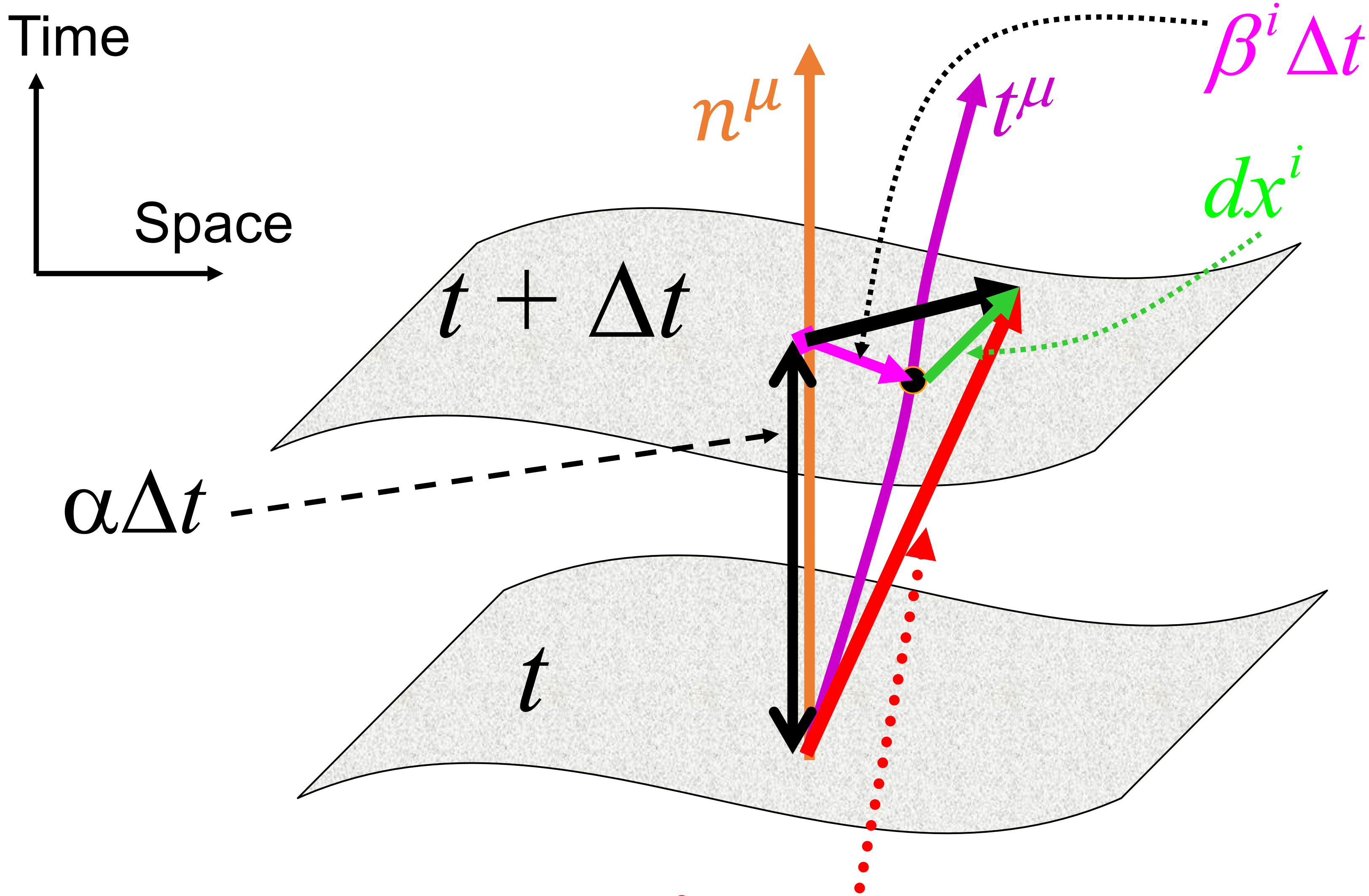
$$K_{ij} := -\gamma_i{}^\mu \gamma_j{}^\nu \nabla_\mu n_\nu = -\frac{1}{2}\mathcal{L}_n \gamma_{ij}$$

Derive this later

Lie derivative wrt n^μ (the $\mathcal{L}_n$ term)

Note: $\mathcal{L}_n Q_{\mu\nu} = n^\alpha \nabla_\alpha Q_{\mu\nu} + Q_{\alpha\mu} \nabla_\nu n^\alpha + Q_{\alpha\nu} \nabla_\mu n^\alpha$
$= n^\alpha \partial_\alpha Q_{\mu\nu} + Q_{\alpha\mu} \partial_\nu n^\alpha + Q_{\alpha\nu} \partial_\mu n^\alpha$

Line element in 3+1 form



$$ds^2 = -(\alpha dt)^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

Structure of variables

$$n^\mu = \left(\frac{1}{\alpha}, \ -\frac{\beta^i}{\alpha}\right), \quad n_\mu = (-\alpha,\ 0); \quad \text{cf. } t^\mu = (1,\ 0)$$

$$n_\mu = -\alpha \nabla_\mu t;$$
$$n^\mu n_\mu = -1$$
$$\alpha n^\mu = t^\mu - \beta^\mu$$

α : lapse function, β^i : shift vector; $\beta_i = \gamma_{ij}\beta^j$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k, \beta_i \\ * \qquad ,\gamma_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2, \beta^i/\alpha^2 \\ * \qquad ,\gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}$$

$$\text{Cf.} \quad \gamma_{\mu\nu} = \begin{pmatrix} \beta^k\beta_k, \beta_i \\ * \qquad ,\gamma_{ij} \end{pmatrix}, \quad \gamma^{\mu\nu} = \begin{pmatrix} 0\ ,0 \\ *\ ,\gamma^{ij} \end{pmatrix}$$

$$K_{\mu\nu} = \begin{pmatrix} K_{ij}\beta^i\beta^j, K_{ij}\beta^j \\ * \qquad ,K_{ij} \end{pmatrix}, \quad K^{\mu\nu} = \begin{pmatrix} 0\ ,0 \\ *\ ,K^{ij} \end{pmatrix}$$

Note: $\gamma_{\mu\nu}n^\mu = 0 = K_{\mu\nu}n^\mu$, but $\gamma_{t\mu}$ & $K_{t\mu}$ are not in general zero

Next step: defining the covariant derivative & K_{ij}

As a first step, it is necessary to define covariant derivative associated with γ_{ij}: D_k

$D_i \gamma_{jk} = 0 \quad \longleftarrow$ required property $\left(\text{cf. } \nabla_\mu g_{\alpha\beta} = 0\right)$

$T^{\alpha\beta\gamma\cdots}{}_{\mu\nu\sigma\cdots}$: Spacetime tensor

Define

$$D_l T^{ijk\cdots}{}_{mno\cdots} = \underbrace{\gamma_l{}^\delta \gamma_\alpha{}^i \gamma_\beta{}^j \gamma_\rho{}^k \cdots \gamma_m{}^\mu \gamma_n{}^\nu \gamma_o{}^\xi}_{\text{projection}} \cdots \nabla_\delta T^{\alpha\beta\rho\cdots}{}_{\mu\nu\xi\cdots}$$

Then,

$$D_l \gamma_{ij} = \gamma_l{}^\delta \gamma_i{}^\alpha \gamma_j{}^\beta \nabla_\delta \gamma_{\alpha\beta} = \gamma_l{}^\delta \gamma_i{}^\alpha \gamma_j{}^\beta \nabla_\delta \left(g_{\alpha\beta} + n_\alpha n_\beta\right)$$

$$= \gamma_l{}^\delta \gamma_i{}^\alpha \gamma_j{}^\beta \nabla_\delta \left(n_\alpha n_\beta\right) = \gamma_l{}^\delta \gamma_i{}^\alpha \gamma_j{}^\beta \left(n_\alpha \nabla_\delta n_\beta + n_\beta \nabla_\delta n_\alpha\right)$$

$$= 0 \quad \left(\because \quad \gamma_{\alpha\beta} n^\beta = 0\right)$$

OK

Relation of K_{ij} with three metric

$$K_{ij} = -\gamma_i^{\alpha}\gamma_j^{\beta}\nabla_\alpha n_\beta = -\frac{1}{2}\gamma_i^{\alpha}\gamma_j^{\beta}\left(\nabla_\alpha n_\beta + \nabla_\beta n_\alpha\right)$$

K_{ij}: symmetric tensor

$$\gamma_i^{\alpha}\gamma_j^{\beta}\nabla_\alpha n_\beta = \left(g_i^{\alpha} + n_i n^{\alpha}\right) g_j^{\beta}\nabla_\alpha n_\beta \qquad \left(n^{\beta}\nabla_\alpha n_\beta = 0\right)$$

$$= \nabla_i n_j + n_i n^{\mu}\nabla_\mu n_j$$

$$\Rightarrow K_{ij} = -\frac{1}{2}\left(\nabla_i n_j + \nabla_j n_i + n^{\mu}\nabla_\mu\left(n_i n_j\right)\right)$$

$n_i n_j = \gamma_{ij} - g_{ij}$

$$= -\frac{1}{2}\left(\gamma_{j\alpha}\nabla_i n^{\alpha} + \gamma_{i\alpha}\nabla_j n^{\alpha} + n^{\mu}\nabla_\mu \gamma_{ij}\right)$$

i.e., $K_{ij} = -\frac{1}{2}\mathcal{L}_n \gamma_{ij}$

$$= -\frac{1}{2}\left(\gamma_{j\alpha}\partial_i n^{\alpha} + \gamma_{i\alpha}\partial_j n^{\alpha} + n^{\mu}\partial_\mu \gamma_{ij}\right)$$

$\alpha n^{\mu} = (1, -\beta^i)$

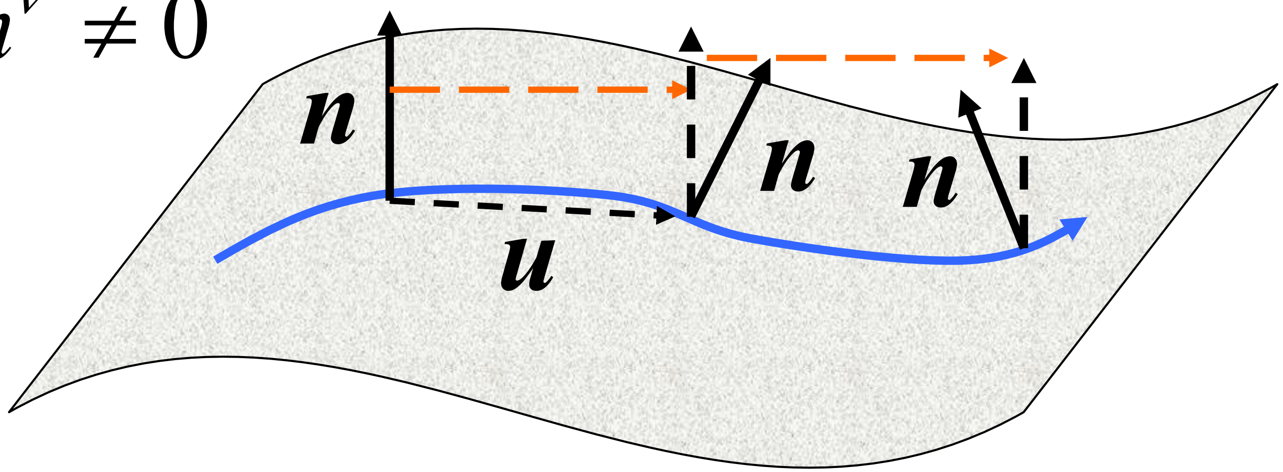
$$= -\frac{1}{2\alpha}\left(\partial_t \gamma_{ij} - D_i\beta_j - D_j\beta_i\right)$$

Note: $\gamma_{j\alpha}\partial_i n^{\alpha} = \alpha^{-1}\gamma_{j\alpha}\partial_i(\alpha n^{\alpha}) = -\alpha^{-1}\gamma_{jk}\partial_i\beta^k$

Geometric meaning of K_{ij}

If space-like hyper-surfaces are curved,
n is not parallel-transported.

$$u^{\mu}\nabla_{\mu}n^{\nu} \neq 0$$



$$u^{i}K_{ij} = -\gamma^{i}_{j}u^{\mu}\nabla_{\mu}n_{i} \neq 0$$

K_{ij} denotes how a chosen spatial hypersurface is curved

Useful, often-used relations

$$K_{\mu\nu} \equiv -\nabla_\mu n_\nu - n_\mu n^\sigma \nabla_\sigma n_\nu = -\nabla_\mu n_\nu - n_\mu D_\nu \ln\alpha$$

$Note:\quad n^\sigma \nabla_\sigma n_\nu = D_\nu \ln\alpha :$ acceleration

We will use these relations for many times in the following

Note also: $K = K_{ij}\gamma^{ij} = -\nabla_\mu n^\mu$

Problem I: Show $n^\alpha \nabla_\alpha n_\mu = D_\mu \ln\alpha$

Last step: 3+1 decomposition of Einstein's equation

$$\begin{cases} G_{\mu\nu} n^\mu n^\nu = 8\pi T_{\mu\nu} n^\mu n^\nu & : \text{Hamiltonian constraint} \\ G_{\mu\nu} n^\mu \gamma^\nu_k = 8\pi T_{\mu\nu} n^\mu \gamma^\nu_k & : \text{Momentum constraint} \end{cases}$$

$$\{\; G_{\mu\nu} \gamma^\mu_i \gamma^\nu_j = 8\pi T_{\mu\nu} \gamma^\mu_i \gamma^\nu_j \quad : \text{Evolution equation}$$

$$g_{\mu\nu} \rightarrow (\gamma_{ij},\ \alpha,\ \beta^k)\ \&\ K_{ij} = -\frac{1}{2}\mathcal{L}_n \gamma_{ij}$$

- First two equations = constraint equations
  - ∵ no second derivative of spatial metric (not hyperbolic, elliptic type equations)
- Last one = evolution equation
  - ∵ hyperbolic equations of spatial metric
- No time derivative for α & β^k : These are gauges

Similar to Maxwell's equations

$$\nabla_i E^i = 4\pi\rho_e$$
$$\nabla_i B^i = 0$$

Constraint equations

$$\partial_t E^i = (\nabla \times B)^i - 4\pi j^i$$
$$\partial_t B^i = -(\nabla \times E)^i$$

Evolution equations
(hyperbolic equations)

<u>Toward the solution:</u>

Step 1: Give an initial condition which satisfies constraints.

Step 2: Solve evolution equations.

All equations have to be written by (γ_{ij}, K_{ij})

Method: Derive Gauss & Codazzi equations

First, derive the Gauss equation

$$\left(D_i D_j - D_j D_i\right)\omega_k = {}^{(3)}R_{ijk}{}^{l}\omega_l;$$

ω_l=spatial vector

Using the definition of 3D covariant derivative,

$$D_i D_j \omega_k = \gamma_i{}^{\alpha}\gamma_j{}^{\beta}\gamma_k{}^{\delta}\nabla_\alpha\left(\gamma_\beta{}^{\mu}\gamma_\delta{}^{\nu}\nabla_\mu\omega_\nu\right)$$

$$= \gamma_i{}^{\alpha}\gamma_j{}^{\beta}\gamma_k{}^{\delta}\nabla_\alpha\nabla_\beta\omega_\delta + \gamma_i{}^{\alpha}\gamma_j{}^{\beta}\gamma_k{}^{\nu}\left(\nabla_\alpha\gamma_\beta{}^{\mu}\right)\left(\nabla_\mu\omega_\nu\right)$$

$$+\gamma_i{}^{\alpha}\gamma_j{}^{\mu}\gamma_k{}^{\delta}\left(\nabla_\alpha\gamma_\delta{}^{\nu}\right)\left(\nabla_\mu\omega_\nu\right)$$

See next page

$$= \gamma_i{}^{\alpha}\gamma_j{}^{\beta}\gamma_k{}^{\delta}\nabla_\alpha\nabla_\beta\omega_\delta - \left(n^{\alpha}\nabla_\alpha\omega_\nu\right)\gamma_k{}^{\nu}K_{ij} - K_{ik}K_j{}^{l}\omega_l$$

$$\therefore \left(D_i D_j - D_j D_i\right)\omega_k = \gamma_i{}^{\alpha}\gamma_j{}^{\beta}\gamma_k{}^{\delta}\left(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right)\omega_\delta$$

$$-\left(K_{ik}K_j{}^{l} - K_{jk}K_i{}^{l}\right)\omega_l$$

where we used

$$\nabla_\alpha \gamma_\beta^{\ \delta} = \nabla_\alpha \left(g_\beta^\delta + n_\beta n^\delta \right) = \nabla_\alpha \left(n_\beta n^\delta \right)$$
$$= n_\beta \nabla_\alpha n^\delta + n^\delta \nabla_\alpha n_\beta$$
$$= -n_\beta \left(K_\alpha^\delta + n_\alpha D^\delta \ln\alpha \right) - n^\delta \left(K_{\alpha\beta} + n_\alpha D_\beta \ln\alpha \right)$$
$$\rightarrow \gamma_i^{\ \alpha} \gamma_j^{\ \beta} \nabla_\alpha \gamma_\beta^{\ \delta} = -n^\delta K_{ij}$$

and

$$\gamma_j^{\ \mu} n^\nu \nabla_\mu \omega_\nu = -\gamma_j^{\ \mu} \omega_\nu \nabla_\mu n^\nu$$
$$= \gamma_j^{\ \mu} \omega_\nu \left(K_\mu^\nu + n_\mu D^\nu \ln\alpha \right) = \omega_k K_j^{\ k}$$

$$n^\mu \omega_\mu = 0$$

Note again: $\nabla_\alpha n_\beta = -K_{\alpha\beta} - n_\alpha D_\beta \ln\alpha$ & $\gamma_{\mu\nu} n^\nu = 0$

Then, we obtain the **Gauss equation**:

$$\left(D_i D_j - D_j D_i\right)\omega_k = {}^{(3)}R_{ijk}{}^{l}\omega_l$$

$$= \gamma_i{}^{\alpha}\gamma_j{}^{\beta}\gamma_k{}^{\delta}R_{\alpha\beta\delta}{}^{l}\omega_l - K_j{}^{l}K_{ik}\omega_l + K_i{}^{l}K_{jk}\omega_l$$

$$\Longrightarrow \quad \boxed{{}^{(3)}R_{ijk}{}^{l} = \gamma_i{}^{\alpha}\gamma_j{}^{\beta}\gamma_k{}^{\delta}R_{\alpha\beta\delta}{}^{l} - K_j{}^{l}K_{ik} + K_i{}^{l}K_{jk}}$$

Note ω_l: arbitrary spatial vector

Contracting the Gauss equation,

$${}^{(3)}R_{ik} = {}^{(3)}R_{ijk}{}^{j} = \gamma_i{}^{\alpha}\gamma_j{}^{\beta}\gamma_k{}^{\delta}R_{\alpha\beta\delta}{}^{j} - K_j{}^{j}K_{ik} + K_i{}^{j}K_{jk}$$

$$= \gamma_i{}^{\alpha}\left(g_j{}^{\beta} + n^{\beta}n_j\right)\gamma_k{}^{\delta}R_{\alpha\beta\delta}{}^{j} - K_j{}^{j}K_{ik} + K_i{}^{j}K_{jk}$$

$$= \gamma_i{}^{\alpha}\gamma_k{}^{\delta}\left(R_{\alpha\delta} + R_{\alpha\beta\delta}{}^{\mu}n^{\beta}n_{\mu}\right) - K_j{}^{j}K_{ik} + K_i{}^{j}K_{jk}$$

Hereafter we write $K = K_j{}^{j}$

Further multiplying γ^{ik}

$$
\begin{aligned}
{}^{(3)}R &= \left(R + 2R_{\alpha\beta}n^{\alpha}n^{\beta}\right) - K^2 + K_{ij}K^{ij} \\
&= 2G_{\alpha\beta}n^{\alpha}n^{\beta} - K^2 + K_{ij}K^{ij} \\
&= 16\pi\frac{G}{c^4}T_{\alpha\beta}n^{\alpha}n^{\beta} - K^2 + K_{ij}K^{ij}
\end{aligned}
$$

Substitute Einstein's equation

$$
\Longrightarrow \boxed{{}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\frac{G}{c^4}T_{\alpha\beta}n^{\alpha}n^{\beta}}
$$

Hamiltonian constraint: 1 component

Note: $G_{\mu\nu}n^{\mu}n^{\nu} = \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)n^{\mu}n^{\nu}$

$$
= R_{\mu\nu}n^{\mu}n^{\nu} + \frac{1}{2}R
$$

Derive Codazzi equation

Start from $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) n^\alpha = -R_{\mu\nu\sigma}{}^\alpha n^\sigma$

$\Longrightarrow \gamma_i{}^\mu \gamma_j{}^\nu \gamma_\alpha^k (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) n^\alpha = -\gamma_i{}^\mu \gamma_j{}^\nu \gamma_\alpha^k R_{\mu\nu\sigma}{}^\alpha n^\sigma$

Now $\gamma_i{}^\mu \gamma_j{}^\nu \gamma_\alpha^k \nabla_\mu \nabla_\nu n^\alpha = \gamma_i{}^\mu \gamma_j{}^\nu \gamma_\alpha^k \nabla_\mu (-K_\nu^\alpha - n_\nu a^\alpha)$

$= -D_i K_j{}^k + a^k K_{ij}$ Note: $a^\alpha = n^\beta \nabla_\beta n^\alpha$

$\therefore \quad \gamma_i{}^\mu \gamma_j{}^\nu \gamma_\alpha^k (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) n^\alpha = -D_i K_j{}^k + D_j K_i{}^k$

$\Longrightarrow \quad D_i K_j{}^k - D_j K_i{}^k = \gamma_i{}^\mu \gamma_j{}^\nu \gamma_\alpha^k R_{\mu\nu\sigma}{}^\alpha n^\sigma$

Codazzi equations

i, k contraction

$\Longrightarrow \quad D_i K_j{}^i - D_j K = \gamma_\alpha^\mu \gamma_j{}^\nu R_{\mu\nu\sigma}{}^\alpha n^\sigma = -\gamma_j{}^\nu R_{\nu\sigma} n^\sigma$

$= -8\pi \frac{G}{c^4} T_{\mu\nu} n^\mu \gamma_j{}^\nu$ Note: $R_{\mu\nu\sigma}{}^\alpha n^\sigma n_\alpha = 0$

Momentum constraint: 3 components

Derive evolution equation I

Start from contracted Gauss equation (see page 24):

$$ {}^{(3)}R_{ij} = \gamma_i{}^{\mu}\gamma_j{}^{\nu}\left(R_{\mu\nu} + R_{\mu\alpha\nu\beta}n^{\alpha}n^{\beta}\right) + K_{jk}K_i{}^{k} - KK_{ij} $$

The term, $\gamma_i{}^{\mu}\gamma_j{}^{\nu}R_{\mu\alpha\nu\beta}n^{\alpha}n^{\beta}$, contains $\partial_t K_{ij}$

To see this, pay attention to

$$ n^{\alpha}\gamma_i{}^{\mu}\gamma_j{}^{\nu}R_{\mu\alpha\nu\beta}n^{\beta} = n^{\alpha}\gamma_i{}^{\mu}\gamma_j{}^{\nu}\left(\nabla_\mu\nabla_\alpha - \nabla_\alpha\nabla_\mu\right)n_\nu $$

Here, $\nabla_\mu\nabla_\alpha n_\nu = \nabla_\mu(-K_{\alpha\nu} - n_\alpha D_\nu \ln\alpha)$

$$ = -\nabla_\mu K_{\alpha\nu} + \left(K_{\mu\alpha} + n_\mu D_\alpha \ln\alpha\right)D_\nu \ln\alpha - n_\alpha \nabla_\mu D_\nu \ln\alpha $$

Thus, $n^{\alpha}\gamma_i{}^{\mu}\gamma_j{}^{\nu}\nabla_\mu\nabla_\alpha n_\nu = -n^{\alpha}\gamma_i{}^{\mu}\gamma_j{}^{\nu}\nabla_\mu K_{\alpha\nu} + D_i D_j \ln\alpha$

And $n^{\alpha}\gamma_i{}^{\mu}\gamma_j{}^{\nu}\nabla_\alpha\nabla_\mu n_\nu = -n^{\alpha}\gamma_i{}^{\mu}\gamma_j{}^{\nu}\nabla_\alpha K_{\mu\nu} - (D_i \ln\alpha) D_j \ln\alpha$

Note again: $\nabla_\alpha n_\beta = -K_{\alpha\beta} - n_\alpha D_\beta \ln\alpha$ & $\gamma_{\mu\nu}n^{\nu} = 0$

Derive evolution equation II

Hence, $n^\alpha \gamma_i{}^\mu \gamma_j{}^\nu R_{\mu\alpha\nu\beta} n^\beta = -n^\alpha \gamma_i{}^\mu \gamma_j{}^\nu \nabla_\mu K_{\alpha\nu} + D_i D_j \ln\alpha$

$$+n^\alpha \gamma_i{}^\mu \gamma_j{}^\nu \nabla_\alpha K_{\mu\nu} + (D_i \ln\alpha)_i D_j \ln\alpha$$

$$= \gamma_i{}^\mu \gamma_j{}^\nu K_{\alpha\nu} \nabla_\mu n^\alpha + \gamma_i{}^\mu \gamma_j{}^\nu n^\alpha \nabla_\alpha K_{\mu\nu} + \frac{1}{\alpha} D_i D_j \alpha$$

Here,

$$\gamma_i{}^\mu \gamma_j{}^\nu K_{\alpha\nu} \nabla_\mu n^\alpha = -K_{\alpha j} \gamma_i{}^\mu \left(K_\mu^\alpha + n_\mu D^\alpha \ln\alpha \right) = -K_{jk} K_i{}^k$$

and $n^\alpha \nabla_\alpha K_{\mu\nu} = \mathcal{L}_n K_{\mu\nu} - K_{\mu\beta} \nabla_\nu n^\beta - K_{\nu\beta} \nabla_\mu n^\beta$

$$= \mathcal{L}_n K_{\mu\nu} + K_{\mu\beta} \left(K_\nu^\beta + n_\nu D^\beta \ln\alpha \right) + K_{\nu\beta} \left(K_\mu^\beta + n_\mu D^\beta \ln\alpha \right)$$

$$\rightarrow \quad \gamma_i{}^\mu \gamma_j{}^\nu n^\alpha \nabla_\alpha K_{\mu\nu} = \mathcal{L}_n K_{ij} + 2 K_{ik} K_j{}^k$$

Therefore, $n^\alpha \gamma_i{}^\mu \gamma_j{}^\nu R_{\mu\alpha\nu\beta} n^\beta = \mathcal{L}_n K_{ij} + K_{ik} K_j{}^k + \frac{1}{\alpha} D_i D_j \alpha$

Derive evolution equation III

Thus, the contracted Gauss equation is written as

$$ {}^{(3)}R_{ij} = \gamma_i{}^{\mu}\gamma_j{}^{\nu}\left(R_{\mu\nu} + R_{\mu\alpha\nu\beta}n^{\alpha}n^{\beta}\right) + K_{jk}K_i{}^{k} - KK_{ij} $$

$$ = \gamma_i{}^{\mu}\gamma_j{}^{\nu}R_{\mu\nu} + \mathcal{L}_n K_{ij} + 2K_{ik}K_j{}^{k} - KK_{ij} + \frac{1}{\alpha}D_i D_j \alpha $$

Here, $\mathcal{L}_n K_{ij} = n^{\alpha}\nabla_{\alpha}K_{ij} + K_{i\mu}\nabla_j n^{\mu} + K_{j\mu}\nabla_i n^{\mu}$

$$ = n^{\alpha}\partial_{\alpha}K_{ij} + K_{i\mu}\partial_j n^{\mu} + K_{j\mu}\partial_i n^{\mu} $$

Note: $K_{\alpha\beta}n^{\beta} = 0$

$$ = \frac{1}{\alpha}\left[\alpha n^{\alpha}\partial_{\alpha}K_{ij} + K_{i\mu}\partial_j(\alpha n^{\mu}) + K_{j\mu}\partial_i(\alpha n^{\mu})\right] $$

Note: $\alpha n^{\beta} = \left(1, -\beta^k\right)$

$$ = \frac{1}{\alpha}\left[(t^{\alpha} - \beta^{\alpha})\partial_{\alpha}K_{ij} - K_{ik}\partial_j\beta^k - K_{jk}\partial_i\beta^k\right] $$

$$ = \frac{1}{\alpha}\left[\partial_t K_{ij} - \beta^k D_k K_{ij} - K_{ik}D_j\beta^k - K_{jk}D_i\beta^k\right] $$

Finally we obtain

Thus, the contracted Gauss equation is written as

$$\partial_t K_{ij} = \alpha\ {}^{(3)}R_{ij} + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{jk} D_i \beta^k$$

$$-2\alpha K_{ik} K_j{}^k + \alpha K K_{ij} - D_i D_j \alpha - \alpha \gamma_i{}^\mu \gamma_j{}^\nu R_{\mu\nu}$$

$$= \alpha\ {}^{(3)}R_{ij} + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{jk} D_i \beta^k$$

$$-2\alpha K_{ik} K_j{}^k + \alpha K K_{ij} - D_i D_j \alpha$$

$$-8\pi \frac{G}{c^4} \alpha \gamma_i{}^\mu \gamma_j{}^\nu \left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right)$$

The evolution equation: 6 components

Note Einstein's equation is written as

$$R_{\mu\nu} = 8\pi \frac{G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right);\ \ T = T_\alpha{}^\alpha$$

Summary of 3+1 formalism

$$
\left.
\begin{aligned}
&{}^{(3)}R - K_{ij}K^{ij} + K^2 = 16\pi T_{\mu\nu} n^\mu n^\nu \\
&D_i K^i_j - D_j K = -8\pi T_{\mu\nu} n^\mu \gamma^\nu_j
\end{aligned}
\right\} \text{Constraints}
$$

$$
\left.
\begin{aligned}
\dot{K}_{ij} &= \alpha\left({}^{(3)}R_{ij} + KK_{ij} - 2K_{il}K^l_j\right) - D_i D_j \alpha \\
&\quad + K_{il} D_j \beta^l + K_{jl} D_i \beta^l + \beta^l D_l K_{ij} \\
&\quad - 8\pi\alpha T_{\mu\nu}\left[\gamma^\mu_i \gamma^\nu_j - \frac{1}{2}(\gamma^{\mu\nu} - n^\mu n^\nu)\gamma_{ij}\right]\frac{G}{c^4} \\
\dot{\gamma}_{ij} &= -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i
\end{aligned}
\right\} \text{Evolution}
$$

$\alpha, \beta^i \;\Leftarrow\;$ Gauge condition

3+1 equations: Constrained system

- 3+1 equations seem to have too many components: Constraints seem to be redundant equations, because γ_{ij} & K_{ij} are determined only by solving evolution eqs.
- NO!
- The constraints are guaranteed to be satisfied *if* the evolution equations are solved precisely
  → **No inconsistency**

Evolution equations of constraints

$$A_{\mu\nu} \equiv G_{\mu\nu} - 8\pi T_{\mu\nu}\frac{G}{c^4}$$

$$= H_0 n_\mu n_\nu + H_i \gamma^i_\mu n_\nu + H_i \gamma^i_\nu n_\mu + H_{ij}\gamma^i_\mu \gamma^i_\nu$$

$H_0 = 0,\ \ H_i = 0 \ \cdots$ H & M Constraints

$H_{ij} = 0 \ \cdots$ Evolution eqs.

$$\nabla_\mu A^\mu_\nu = 0 \quad \Rightarrow$$

$$\left(\partial_t - \beta^l \partial_l\right) H_0 = \alpha K H_0 - 2H_i D^i \alpha - \alpha D_i H^i + \alpha H_{ij} K^{ij}$$

$$\left(\partial_t - \beta^l \partial_l\right) H_i = -H_0 D_i \alpha + \alpha K H_i + H_k \beta^k{}_{,i} - D_k\left(\alpha H^k_i\right)$$

Problem II: derive these equations

If constrains are zero at $t = 0$ and the evolution equations are satisfied for any t, the **constraints are always satisfied.**

Counting the degree of freedom

- γ_{ij} & K_{ij} have in total 12 components
- Real dynamical freedom?
- 4 components are constrained by the presence of constraint equations
- 4 components correspond to the gauge freedom
- Remaining degree of the freedom is 12–4–4=4
- 2 for each of 3 metric and extrinsic curvature
  → This is the dynamical degree of the freedom (i.e., gravitational waves)
- In the alternative theories of gravity, this situation is usually changed

Nature of standard 3+1 formalism

- Evolution equations are hyperbolic equations of 6 components, but *it is not simple wave equation*
- Unpleasant additional terms even in the linear order

$$\dot{K}_{ij} \sim -\frac{1}{2}\ddot{\gamma}_{ij}$$

$$^{(3)}R_{ij} \sim \frac{1}{2}\left[-\Delta\gamma_{ij} + \underbrace{\gamma^{kl}\left(\gamma_{ik,lj} + \gamma_{jk,li} - \partial_{ij}\gamma_{kl}\right)}\right]$$

$$\Rightarrow \quad \ddot{\gamma}_{ij} \approx \Delta\gamma_{ij} + \cdots$$

$cf.$ Maxwell's equation

$$\partial_\alpha\partial^\alpha A_\beta \underline{-\partial_\beta\partial_\alpha A^\alpha} = 0$$

Linearized vacuum equations

Linearized Einstein equations

with $\alpha=1$ & $\beta^i=0$ Just for simplicity

$\gamma_{ij}=\delta_{ij}+h_{ij};\ \ |h_{ij}|\ll 1$

Evolution eq. : $\ddot{h}_{ij}=\Delta h_{ij}-h_{ik,kj}-h_{jk,ki}+h_{kk,ij}$

These terms cause a problem in numerical simulation

Constraint H : $\Delta h_{ii}-h_{ik,ki}=0$

M : $\dot{h}_{ij,i}-\dot{h}_{ii,j}=0$

Problem III: Derive the linear equations for $\alpha\neq 1$ & $\beta^k\neq 0$

Problem IV: Find a gauge condition which leads to $\ddot{h}_{ij}=\Delta h_{ij}$

Stability analysis

Decomposition:

$$h_{ij} = A\delta_{ij} + C_{,ij} + 2B_{(i,j)} + h_{ij}^{\mathrm{TT}}$$

$A,\ C$: scalar, B_i : vector, h_{ij}^{TT} : tensor

definition: $B_{i,i} = 0, \quad h_{ij,j}^{\mathrm{TT}} = h_{ii}^{\mathrm{TT}} = 0$

$\Rightarrow$ Trace $\quad h_{ii} = 3A + \Delta C,$

Divergence $\ h_{ij,j} = A_{,i} + \Delta(C_{,i} + B_i)$

$$\xrightarrow{\text{substitute}} \quad \ddot{h}_{ij} = \Delta h_{ij} - h_{ik,kj} - h_{jk,ki} + h_{kk,ij}$$

$$\Delta h_{ii} - h_{ik,ki} = 0, \quad \dot{h}_{ij,i} - \dot{h}_{ii,j} = 0$$

Substitute the decomposition form into these

Linearized 3+1 equations with α=1 & β^i=0

$$\text{Constraints}: \begin{cases} \text{H:} \quad \Delta A = 0 \\ \text{M:} \ \partial_t \left(-2A_{,i} + \Delta B_i \right) = 0 \end{cases}$$

$$\text{Evolution eqs.}: \begin{cases} \ddot{h}_{ij}^{\mathrm{TT}} = \Delta h_{ij}^{\mathrm{TT}} \\ \ddot{B}_{(i,j)} = 0 \\ \ddot{A} = \Delta A \\ \ddot{C} = A \end{cases}$$

Wave equation:
fine

Strange forms;
Locally determined
→ problem

Solutions I

1 Equations for h_{ij}^{TT} =Wave equations

$\Rightarrow$ True degree of GWs: No problem

2 Constraint (H) : $A = 0$

& Evolution equation for $A =$ wave eq.

$\Rightarrow$ Violated constraint will propagate away.

3 Constraint (M) : $\left(-2A_{,i} + \Delta B_i\right) = F\left(x^i\right)$

For $A = 0$, $B_i = F_{B_i}\left(x^i\right)$

Numerical integration of zero is zero

Evolution equation for B_i gives $\dot{B}_i = F_B\left(x^i\right) \to 0$

& $B_i = \left\{F_B\left(x^i\right)t\right\} + F_{B_i}\left(x^i\right)$

No problem if the constraint is satisfied

Solutions II

C is not constrainted by constraints,

but determined by $\ddot{C} = A,\ \ddot{A} = \Delta A$

If constraint is violated and $A \neq 0$ initially,

$$A = \sum_{l,m} Y_{lm} \frac{\ddot{f}_{lm}(r-t) + \ddot{g}(r+t)}{r}$$

$$\Rightarrow C = \sum Y_{lm} \frac{f(r-t) + g(r+t)}{r} + C_1 t + C_2$$

A small constraint violation results in a serious problem

To summarize

- In the original 3+1 (N+1) formalism, *if constraints are violated even slightly*, the error increases with time *even in a nearly flat spacetime* with no limit
- That is, **it is unsuitable for numerical relativity**
- **The origin of the problem:**

$$\ddot{h}_{ij} = \Delta h_{ij} \underbrace{- h_{ik,kj} - h_{jk,ki} + h_{kk,ij}}$$

First, noticed by Takashi Nakamura (1987)

Section III: BSSN formalism

Essence

- **Need reformulation of 3+1 formalism**
- At least, in the linear level, wave equations must be derived

Define new variables

$$\begin{cases} F_i = h_{ij,j} \\ \Phi = h_{ii} \end{cases}$$

F_i and Φ are considered as independent variables
→ h_{ij} looks obeying a wave equation

and rewrite as

$$\ddot{h}_{ij} = \Delta h_{ij} - F_{i,j} - F_{j,i} + \Phi_{,ij}$$

How to derive the evolution equation for F_i & Φ ?

Reformulation using constraint equations

Momentum constraint: $\dot{h}_{ij,j} - \dot{h}_{jj,i} = 0$

$\Rightarrow \quad \dot{F}_i - \dot{\Phi}_{,i} = 0$: Evolution eq for F_i

Trace of $\ddot{h}_{ij} = \Delta h_{ij} - F_{i,j} - F_{j,i} + \Phi_{,ij}$

$\Rightarrow \quad \ddot{\Phi} = 2\Delta\Phi - 2F_{i,i}$

Hamiltonian constraint: $\Delta\Phi - F_{i,i} = 0$

$\Rightarrow \quad \ddot{\Phi} = 0 \quad \Rightarrow \quad \dot{\Phi} = 0 \quad \Rightarrow \quad \dot{F}_i = 0$

$\Rightarrow \quad \ddot{h}_{ij} = \Delta h_{ij}$

Using the constraint equations appropriately in the reformulation, we can guarantee the hyperbolicity

Similar definition of new variables F_i & Φ should be possible even in the non-linear case

↓

BSSN formalism

First, derived by T. Nakamura (1987); essential idea was described in his original paper.
Subsequently modified by Shibata & Nakamura (1995), Baumgarte & Shapiro (1998) but no qualitative modification

Original version of the BSSN formalism

(Shibata-Nakamura 1995)

First of all, write the line element

$$ds^2 = -\left(\alpha^2 - \beta_i \beta^i\right) dt^2 + 2\beta_i dx^i dt + e^{4\phi} \tilde{\gamma}_{ij} dx^i dx^j$$

Here, $\det\left(\tilde{\gamma}_{ij}\right) = 1.$ (ϕ corresponds to Φ.)

As conjugates for $\tilde{\gamma}_{ij}$ and ϕ, define

$$\tilde{A}_{ij} = e^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) \text{ and } K = \mathrm{trace}\left(K_{ij}\right) = \gamma^{ij}K_{ij}$$

Up to here, we increase 2 variables (ϕ, K) and two new constraints, $\det\left(\tilde{\gamma}_{ij}\right) = 1$ and $\tilde{A}_{ij}\tilde{\gamma}_{ij} = 0$

Then, the equations are

$$(\partial_t - \beta^l \partial_l)\tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{il}\beta^l{}_{,j} + \tilde{\gamma}_{jl}\beta^l{}_{,i} - \frac{2}{3}\tilde{\gamma}_{ij}\beta^l{}_{,l}$$

$$(\partial_t - \beta^l \partial_l)\phi = \frac{1}{6}\left(-\alpha K + \beta^l{}_{,l}\right)$$

$$(\partial_t - \beta^l \partial_l)\tilde{A}_{ij} = \alpha e^{-4\phi}\left(R_{ij} - \frac{1}{3}\gamma_{ij}R\right) - e^{-4\phi}\left(D_i D_j \alpha - \frac{1}{3}\gamma_{ij}\Delta\alpha\right)$$

$$+\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) + \tilde{A}_{il}\partial_j\beta^l + \tilde{A}_{jl}\partial_i\beta^l - \frac{2}{3}\beta^l{}_{,l}\tilde{A}_{ij}$$

$$-8\pi\alpha e^{-4\phi}T_{\mu\nu}\left[\gamma_i^\mu\gamma_j^\nu - \frac{1}{3}\gamma^{\mu\nu}\gamma_{ij}\right]$$

H-constraint is used

$$(\partial_t - \beta^l \partial_l)K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) - \Delta\alpha + 4\pi\alpha T_{\mu\nu}\left(n^\mu n^\nu + \gamma^{\mu\nu}\right)$$

Note $\frac{G}{c^4} = 1$

Not sufficient in this stage !

Note: don't misunderstand the essence!

- The linear analysis for the simple conformally-decomposed formalism shows that
  "System is even more unstable"
- **An *exponentially growing* mode appear**

$$\ddot{h}_{ij} = \Delta h_{ij} - h_{ik,kj} - h_{jk,ki}$$

$$h_{ij} = C_{,ij} + B_{(i,j)} + h_{ij}^{\mathrm{TT}}$$

$$\Rightarrow \quad \ddot{C} = -\Delta C$$

$h_{kk,ij} \rightarrow \phi_{,ij}$ was already done

"Conformal formalism" is a crazy naming for BSSN formulation

Linear analysis shows that the problems come from R_{ij}

$R_{ij} = \tilde{R}_{ij} + R^{\phi}_{ij}$: $\tilde{R}_{ij}$ is Ricci tensor of $\tilde{\gamma}_{ij}$

$$R^{\phi}_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{\gamma}_{ij}\tilde{D}_k\tilde{D}^k\phi + 4\left(\tilde{D}_i\phi\right)\tilde{D}_j\phi - 4\tilde{\gamma}_{ij}\left(\tilde{D}_k\phi\right)\tilde{D}^k\phi \quad \cdots \text{ scalar part}$$

$$\tilde{R}_{ij} = \frac{1}{2}\left[-\tilde{\gamma}^{kl}\left(\tilde{\gamma}_{ij,kl} - \tilde{\gamma}_{ik,jl} - \tilde{\gamma}_{jk,il}\right)\right] + \tilde{\gamma}^{kl}{}_{,k}\tilde{\Gamma}_{l,ij} - \tilde{\Gamma}^l_{jk}\tilde{\Gamma}^k_{il}$$

See appendix I for R^{ϕ}_{ij}
or textbook by Wald (1984)

$\tilde{\gamma}_{kk,ij}$ was already rewritten by $\phi_{,ij}$

Write formally $\tilde{\gamma}^{ij} = \delta^{ij} + f^{ij}$

$$\Rightarrow \tilde{\gamma}^{kl}\left(\tilde{\gamma}_{ij,kl} - \tilde{\gamma}_{ik,jl} - \tilde{\gamma}_{jk,il}\right)$$

$$= \Delta_{\text{flat}}\tilde{\gamma}_{ij} - \delta^{kl}\left(\tilde{\gamma}_{ik,jl} + \tilde{\gamma}_{jk,il}\right)$$ ← Linear

$$+ f^{kl}\left(\tilde{\gamma}_{ij,kl} - \tilde{\gamma}_{ik,jl} - \tilde{\gamma}_{jk,il}\right)$$ ← Nonlinear

$\Rightarrow$ Define $F_i = \delta^{kl}\tilde{\gamma}_{ik,l}$

as in the linear case

$$\Rightarrow \Delta_{\text{flat}}\tilde{\gamma}_{ij} - \delta^{kl}\left(\tilde{\gamma}_{ik,jl} + \tilde{\gamma}_{jk,il}\right)$$

$$= \Delta_{\text{flat}}\tilde{\gamma}_{ij} - \left(F_{i,j} + F_{j,i}\right)$$

cf. page 42

Next step: Derive evolution equations for the new variable F_i *using the momentum constraint*

- As in the linear case, the equation equation for F_i should be derived from momentum constraint

Momentum constraint:

$$\tilde{D}_i\left(e^{6\phi}\tilde{A}^i_j\right)-\frac{2}{3}e^{6\phi}\tilde{D}_jK=8\pi J_j e^{6\phi}$$

or $$\tilde{D}_i\left(\alpha\tilde{A}^i_j\right)+\left\{6\alpha\left(\tilde{D}_i\phi\right)-\tilde{D}_i\alpha\right\}\tilde{A}^i_j-\frac{2}{3}\alpha\tilde{D}_jK=8\pi\alpha J_j$$

Here, $$2\alpha\tilde{A}_{ij}=-(\partial_t-\beta^l\partial_l)\tilde{\gamma}_{ij}+\tilde{\gamma}_{il}\beta^l{}_{,j}+\tilde{\gamma}_{jl}\beta^l{}_{,i}-\frac{2}{3}\tilde{\gamma}_{ij}\beta^l{}_{,l}$$

Thus, a term $\tilde{D}_i\left(\tilde{\gamma}^{ik}\partial_t\tilde{\gamma}_{jk}\right)$ appears and

$$\tilde{D}_i\left(\tilde{\gamma}^{ik}\partial_t\tilde{\gamma}_{jk}\right)=\tilde{\gamma}^{ik}\left(\partial_i\partial_t\tilde{\gamma}_{jk}-\tilde{\Gamma}^l_{ij}\partial_t\tilde{\gamma}_{kl}-\tilde{\Gamma}^l_{ik}\partial_t\tilde{\gamma}_{jl}\right)$$

$$=\partial_t F_j+f^{ik}\partial_i\partial_t\tilde{\gamma}_{jk}-\tilde{\gamma}^{ik}\left(\tilde{\Gamma}^l_{ij}\partial_t\tilde{\gamma}_{kl}+\tilde{\Gamma}^l_{ik}\partial_t\tilde{\gamma}_{jl}\right)$$

In other view, *the momentum constraint can be considered as the evolution equation for F_i*

Other terms with $\partial_t\tilde{\gamma}_{jk}$ are rewritten using

$$(\partial_t-\beta^l\partial_l)\tilde{\gamma}_{ij}=-2\alpha\tilde{A}_{ij}+\tilde{\gamma}_{il}\beta^l{}_{,j}+\tilde{\gamma}_{jl}\beta^l{}_{,i}-\frac{2}{3}\tilde{\gamma}_{ij}\beta^l{}_{,l}$$

Equation for F_i

$$\left(\partial_t - \beta^k \partial_k\right) F_i$$

$$= 2\alpha\left(f^{jk}\tilde{A}_{ij,k} + \tilde{\gamma}^{jk}{}_{,k}\tilde{A}_{ij} - \frac{1}{2}\tilde{A}^{jk}\tilde{\gamma}_{jk,i} + 6\phi_{,k}\tilde{A}_i^k - \frac{2}{3}K_{,i} \right)$$

$$-2\delta^{jk}\alpha_{,k}\tilde{A}_{ij} + \delta^{jl}\beta^k{}_{,l}\tilde{\gamma}_{ij,k}$$

$$+\left(\tilde{\gamma}_{ik}\beta^k{}_{,j} + \tilde{\gamma}_{jk}\beta^k{}_{,i} - \frac{2}{3}\tilde{\gamma}_{ij}\beta^k{}_{,k} \right)_{,l}\delta^{jl} - 16\pi\alpha J_i$$

Note no nonlinear term of $\left(h_{ij}, \tilde{A}_{ij}\right)$; $h_{ij} = \tilde{\gamma}_{ij} - \delta_{ij}$

This is an alternative form of the momentum constraint

Summary of BSSN formalism

- Definition of 3 additional variables (5 components) (F_i, K, ϕ) is essential, in particular F_i is the key
- **Conformal transformation is *not* essential at all:** Only with conformal transformation, the resulting formalism does not work (even worse than original)
  → "Conformal formalism" is a too bad naming.
- The increase of the new variables results in the increase of new constraints: **17 evolution equations with 9 constraint equation.**
- **Momentum constraint is solved as an evolution equation**

Alternative (Baumgarte-Shapiro ,1998)

Define $\Gamma^i = -\tilde{\gamma}^{ij}{}_{,j}$ instead of $F_i = \delta^{jk}\tilde{\gamma}_{ij,k}$.

(In the linear level, both reduce to $h_{ij,j}$.)

$$\left(\partial_t - \beta^k \partial_k\right)\Gamma^i = 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} - \frac{2}{3}\tilde{\gamma}^{ij}K_{,j} + 6\phi_{,j}\tilde{A}^{ij}\right)$$

$$-\tilde{\Gamma}^j\beta^i_{,j} + \frac{2}{3}\tilde{\Gamma}^i\beta^j{}_{,j} + \tilde{\gamma}^{jk}\beta^i{}_{,jk} + \frac{1}{3}\tilde{\gamma}^{ik}\beta^j{}_{,jk}$$

$$-16\pi\alpha\tilde{\gamma}^{ij}J_j$$

Slightly simpler, but essentially no difference from original

$$\tilde{R}_{ij} = -\frac{1}{2}\left(\tilde{\gamma}^{kl}\tilde{\gamma}_{ij,kl} \underline{- \tilde{\gamma}_{ik}\Gamma^k{}_{,j} - \tilde{\gamma}_{jk}\Gamma^k{}_{,i}}\right)$$

$$-\frac{1}{2}\left(\tilde{\gamma}^{kl}{}_{,j}\tilde{\gamma}_{ik,l} + \tilde{\gamma}_{jk,l}\tilde{\gamma}^{kl}{}_{,i} - \tilde{\gamma}_{ij,k}\Gamma^k\right) - \tilde{\Gamma}^l_{ik}\tilde{\Gamma}^k_{jl}$$

Nine components of constraints

Hamiltonian constraint (1)

$$ {}^{(3)}R - K_{ij}K^{ij} + K^2 = 16\pi T_{\mu\nu} n^\mu n^\nu $$

Momentum constraint (3)

$$ D_i K^i_j - D_j K = -8\pi T_{\mu\nu} n^\mu \gamma^\nu_j $$

Tracefree condition for $\tilde{A}_{ij}$ (1)

$$ \tilde{A}_{ij} \tilde{\gamma}^{ij} = 0 $$

Determinant=1 for $\tilde{\gamma}_{ij}$ (1)

$$ \det\left(\tilde{\gamma}_{ij} \right) = 1 $$

Auxiliary variable (3)

$$ F_i = \tilde{\gamma}_{ij,j} \quad \text{or} \quad \Gamma^i = -\tilde{\gamma}^{ij}{}_{,j} $$

Puncture-BSSN

(Campanelli et al. 2005)

Define $\chi = e^{-4\phi}$ (or $W = e^{-2\phi}$) instead of ϕ to follow a black hole spacetime.

Schwarzschild spacetime in the isotropic coordinates:

$$ds^2 = -\left(\frac{1 - M/2r}{1 + M/2r}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dx^2 + dy^2 + dz^2\right)$$

$$\phi = \ln\left(1 + \frac{M}{2r}\right) \xrightarrow{r \to 0} \infty$$

Note: r=0 is nothing but a coordinate singularity

Define $\chi = \left(1 + \frac{M}{2r}\right)^{-4}$: regular everywhere

$\Rightarrow$ BH spacetime can be numerically followed with no special technique

With new conformal factor

$$\chi = e^{-4\phi} \quad \text{or} \quad W = e^{-2\phi}$$

χ, W are not divergent

$$(\partial_t - \beta^l \partial_l)\phi = \frac{1}{6}\left(-\alpha K + \beta^l{}_{,l}\right)$$

$$\Rightarrow \quad (\partial_t - \beta^l \partial_l) W = \frac{W}{3}\left(\alpha K - \beta^l{}_{,l}\right)$$

$$\psi^{-4} R_{ij}^{\psi}$$

$$= 2e^{-4\phi}\left[-\tilde{D}_i \tilde{D}_j \phi - \tilde{\gamma}_{ij}\tilde{\Delta}\phi + 2\tilde{D}_i\phi \tilde{D}_j\phi - 2\tilde{\gamma}_{ij}\tilde{D}_k\phi\tilde{D}^k\phi\right]$$

$$\Rightarrow \quad \psi^{-4} R_{ij}^{\psi} = W\tilde{D}_i\tilde{D}_j W + \tilde{\gamma}_{ij}\left(W\tilde{\Delta}W - 2\tilde{D}_k W\tilde{D}^k W\right)$$

No divergent term even for BH spacetime:

Coordinate singularities are well handled

Extension to N+1 case ($N > 3$): straightforward

- Structure of the equations are unchanged
- BSSN formalism is slightly modified because the dimension is different (Yoshino-Shibata '09)
- In the following, spacetime dimension is denoted by $D = N+1$

For $ds^2 = -\left(\alpha^2 - \beta_k \beta^k\right) dt^2 + 2\beta_k dx^k dt + \chi^{-1} \tilde{\gamma}_{ij} dx^i dx^j$

$$(\partial_t - \beta^l \partial_l)\tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{il}\beta^l{}_{,j} + \tilde{\gamma}_{jl}\beta^l{}_{,i} - \frac{2}{D-1}\tilde{\gamma}_{ij}\beta^l{}_{,l}$$

$$(\partial_t - \beta^l \partial_l)\chi = \frac{2\chi}{D-1}\left(\alpha K - \beta^l{}_{,l}\right)$$

$$\begin{aligned}(\partial_t - \beta^l \partial_l)\tilde{A}_{ij} = {} & \alpha\chi\left(R_{ij} - \frac{1}{D-1}\gamma_{ij} R\right) - \chi\left(D_i D_j \alpha - \frac{1}{D-1}\gamma_{ij}\Delta\alpha\right) \\ & + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) + \tilde{A}_{il}\partial_j \beta^l + \tilde{A}_{jl}\partial_i \beta^l - \frac{2}{D-1}\beta^l{}_{,l}\tilde{A}_{ij} \\ & - 8\pi\alpha\chi T_{\mu\nu}\left[\gamma^\mu_i \gamma^\nu_j - \frac{1}{D-1}\gamma^{\mu\nu}\gamma_{ij}\right]\end{aligned}$$

$$(\partial_t - \beta^l \partial_l)K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{K^2}{D-1}\right) - \Delta\alpha + \frac{8\pi\alpha}{D-2} T_{\mu\nu}\left(\left(D-3\right) n^\mu n^\nu + \gamma^{\mu\nu}\right)$$

$$\left(\partial_t - \beta^l \partial_l\right)\Gamma^i = 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} - \frac{D-2}{D-1}\tilde{\gamma}^{ij}K_{,j} - \frac{D-1}{2\chi}\chi_{,j}\tilde{A}^{ij}\right)$$

$$-\tilde{\Gamma}^j\beta^i_{,j} + \frac{2}{D-1}\tilde{\Gamma}^i\beta^j_{,j} + \tilde{\gamma}^{jk}\beta^i_{,jk} + \frac{D-3}{D-1}\tilde{\gamma}^{ik}\beta^j_{,jk}$$

$$-16\pi\alpha\tilde{\gamma}^{ij}J_j$$

$$R_{ij} = \tilde{R}_{ij} + R^\chi_{ij}$$

$$R^\chi_{ij} = \frac{D-3}{2\chi}\tilde{D}_i\tilde{D}_i\chi + \frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{D}_k\tilde{D}^k\chi - \frac{D-3}{4\chi^2}\left(\tilde{D}_i\chi\right)\tilde{D}_j\chi$$

$$-\frac{D-1}{4\chi^2}\tilde{\gamma}_{ij}\left(\tilde{D}_k\chi\right)\tilde{D}^k\chi$$

Robust for any dimension (at least up to 7D, Shibata & Yoshino, '10)

Appendix I: Conformal decomposition 1

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}, \quad D_i \gamma_{jk} = 0, \quad \tilde{D}_i \tilde{\gamma}_{jk} = 0$$

$$\Gamma^i_{jk} = \frac{1}{2} \gamma^{il} \left(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk} \right)$$

$$= \frac{1}{2} \tilde{\gamma}^{il} \left(\partial_j \tilde{\gamma}_{kl} + \partial_k \tilde{\gamma}_{jl} - \partial_l \tilde{\gamma}_{jk} \right)$$

$$+ \frac{2}{\psi} \tilde{\gamma}^{il} \left(\tilde{\gamma}_{kl} \partial_j \psi + \tilde{\gamma}_{jl} \partial_k \psi - \tilde{\gamma}_{jk} \partial_l \psi \right)$$

$$= \tilde{\Gamma}^i_{jk} + \frac{2}{\psi} \left(\delta^i_k \tilde{D}_j \psi + \delta^i_j \tilde{D}_k \psi - \tilde{\gamma}_{jk} \tilde{D}^i \psi \right)$$

$$= \tilde{\Gamma}^i_{jk} + C^i_{jk}$$

Conformal decomposition 2

$$R_{jk} = \partial_i \Gamma^i_{jk} - \partial_j \Gamma^i_{ik} + \Gamma^i_{jk}\Gamma^l_{il} - \Gamma^i_{jl}\Gamma^l_{ik}$$

$$= \partial_i \tilde{\Gamma}^i_{jk} - \partial_j \tilde{\Gamma}^i_{ik} + \tilde{\Gamma}^i_{jk}\tilde{\Gamma}^l_{il} - \tilde{\Gamma}^i_{jl}\tilde{\Gamma}^l_{ik}$$

$$+ \partial_i C^i_{jk} - \partial_j C^i_{ik} + C^i_{jk} C^l_{il} - C^i_{jl} C^l_{ik}$$

$$+ \tilde{\Gamma}^i_{jk} C^l_{il} - \tilde{\Gamma}^i_{jl} C^l_{ik} + C^i_{jk}\tilde{\Gamma}^l_{il} - C^i_{jl}\tilde{\Gamma}^l_{ik}$$

$$= \tilde{R}_{jk} + \tilde{D}_i C^i_{jk} - \tilde{D}_j C^i_{ik} + C^i_{jk} C^l_{il} - C^i_{jl} C^l_{ik}$$

$$R^\psi_{jk} = \tilde{D}_i C^i_{jk} - \tilde{D}_j C^i_{ik} + C^i_{jk} C^l_{il} - C^i_{jl} C^l_{ik}$$

$$= \frac{1}{\psi^2}\left(6\tilde{D}_j\psi\tilde{D}_k\psi - 2\tilde{\gamma}_{jk}\tilde{D}_l\psi\tilde{D}^l\psi\right)$$

$$-\frac{2}{\psi}\left(\tilde{D}_j\tilde{D}_k\psi + \tilde{\gamma}_{jk}\tilde{\Delta}\psi\right)$$

Appendix II

How to derive the evolution equations for constraints

$$A_{\mu\nu} := G_{\mu\nu} - 8\pi \frac{G}{c^4} T_{\mu\nu}$$
$$= H_0 n_\mu n_\nu + H_i \gamma^i{}_\mu n_\nu + H_i \gamma^i{}_\nu n_\mu + H_{ij}$$

where $H_0 = A_{\mu\nu} n^\mu n^\nu$, $H_i = -A_{\mu\nu} n^\mu \gamma^\nu{}_i$, $H_{ij} = A_{\mu\nu} \gamma^\mu{}_i \gamma^\nu{}_j$

$H_0 = 0$, $H_i = 0$: Hamiltonian, momentum constraints

$H_{ij} = 0$: evolution equation

$\nabla_\mu A^\mu{}_\nu = 0$: Bianchi identity $\nabla_\mu G^\mu{}_\nu = 0$ & EOM, $\nabla_\mu T^\mu{}_\nu = 0$

In the following we often use

$$A^\mu{}_\nu n^\nu = -H_0 n^\mu - H^\mu, \qquad H^\mu n_\mu = 0$$

Evolution equation of the Hamiltonian constraint

$$0 = n^\nu \nabla_\mu A^\mu{}_\nu = \nabla_\mu\left(A^\mu{}_\nu n^\nu\right) - A^{\mu\nu}\nabla_\mu n_\nu$$

$$= -\nabla_\mu(H_0 n^\mu + H^\mu) + A^{\mu\nu}\left(K_{\mu\nu} + n_\mu D_\nu \ln\alpha\right)$$

$$= -n^\mu \nabla_\mu H_0 - H_0 \nabla_\mu n^\mu - \nabla_\mu H^\mu + H^{ij}K_{ij} - H^k D_k \ln\alpha$$

$$= -\frac{1}{\alpha}\left(\partial_t - \beta^k\partial_k\right)H_0 + H_0 K - \frac{1}{\alpha}D_k\left(\alpha H^k\right) + H^{ij}K_{ij} - H^k D_k \ln\alpha$$

$$\Longrightarrow \left(\partial_t - \beta^k\partial_k\right)H_0 = \alpha H_0 K - \frac{1}{\alpha}D_k\left(\alpha^2 H^k\right) + \alpha H^{ij}K_{ij}$$

where we used $H^\mu n_\mu = 0$, $\nabla_\mu H^\mu = \alpha^{-1}D_k\left(\alpha H^k\right)$,

$$\nabla_\mu n_\nu = -\left(K_{\mu\nu} + n_\mu D_\nu \ln\alpha\right)$$

Evolution of the momentum constraint

$$0 = \nabla_\mu A^\mu{}_i = \frac{1}{\alpha\sqrt{\gamma}}\partial_\mu\left(\alpha\sqrt{\gamma}A^\mu{}_i\right) - \frac{1}{2}A^{\mu\nu}\partial_i g_{\mu\nu}$$

Now, we have

$$A^\mu{}_i = A^\mu{}_\nu \gamma^\nu{}_i = H_i n^\mu + H^\mu{}_i$$

$$\frac{1}{2}A^{\mu\nu}\partial_i g_{\mu\nu} = -H_0\partial_i \ln\alpha + \frac{1}{\alpha}H_k\partial_i\beta^k + \frac{1}{2}H^{jk}\partial_i\gamma_{jk}$$

Hence, $0 = \frac{1}{\sqrt{\gamma}}\partial_\mu\left[\alpha\sqrt{\gamma}\left(H_i n^\mu + H^\mu{}_i\right)\right]$

$$-\alpha\left(-H_0\partial_i \ln\alpha + \frac{1}{\alpha}H_k\partial_i\beta^k + \frac{1}{2}H^{jk}\partial_i\gamma_{jk}\right)$$

$$= \alpha n^\mu \partial_\mu H_i + \frac{H_i}{\sqrt{\gamma}}\partial_\mu[\alpha\sqrt{\gamma}n^\mu] + \frac{1}{\sqrt{\gamma}}\partial_\mu\left[\alpha\sqrt{\gamma}H^\mu{}_i\right]$$

$$-\left(-H_0\partial_i\alpha + H_k\partial_i\beta^k + \frac{\alpha}{2}H^{jk}\partial_i\gamma_{jk}\right)$$

Continued

Then, $0 = \alpha n^\mu \partial_\mu H_i + \alpha H_i \nabla_\mu n^\mu + \frac{1}{\sqrt{\gamma}}\partial_k\left(\alpha\sqrt{\gamma}H^k{}_i\right)$

$$- \frac{\alpha}{2} H^{jk}\partial_i \gamma_{jk} + H_0 D_i \alpha - H_k \partial_i \beta^k$$

$$= \left(\partial_t - \beta^k \partial_k\right) H_i - \alpha K H_i + D_k\left(\alpha H^k{}_i\right) + H_0 D_i \alpha - H_k \partial_i \beta^k$$

$$= \partial_t H_i - \alpha K H_i + D_k\left(\alpha H^k{}_i\right) + H_0 D_i \alpha - \left(\beta^k D_k H_i + H_k D_i \beta^k\right)$$

Thus,

$$\partial_t H_i = \alpha K H_i - D_k\left(\alpha H^k{}_i\right) - H_0 D_i \alpha + \left(\beta^k D_k H_i + H_k D_i \beta^k\right)$$

Or

$$\left(\partial_t - \beta^k \partial_k\right) H_i = \alpha K H_i - D_k\left(\alpha H^k{}_i\right) - H_0 D_i \alpha + \beta^k \partial_k H_i$$

Appendix III

Vacuum linearized Einstein equation with arbitrary gauge

$$\alpha = 1 - \frac{a}{2} \ (|a| \ll 1), \qquad \beta^i \neq 0, \qquad \gamma_{ij} = \delta_{ij} + h_{ij}$$

Evolution equation:

$$\ddot{h}_{ij} = \Delta h_{ij} - h_{ik,kj} - h_{jk,ki} + h_{kk,ij} - a_{,ij} + \dot{\beta}_{i,j} + \dot{\beta}_{j,i}$$

Hamiltonian constraint: $\Delta h_{kk} - k_{ik,ki} = 0$

Momentum constraint: $\dot{h}_{ik,k} - \dot{h}_{kk,i} - \left(\beta_{i,kk} - \beta_{k,ki}\right) = 0$

Harmonic gauge

$$\partial_\mu(\sqrt{-g}\,g^{\mu\nu}) = 0$$

$$\Longrightarrow \begin{cases} -\dot{a} - \dot{h}_{kk} + 2\beta_{k,k} = 0 \\ \dot{\beta}_i - h_{ik,k} - \dfrac{1}{2}\left(a_{,i} - h_{kk,i}\right) = 0 \end{cases}$$

$$\Longrightarrow -h_{ik,kj} - h_{jk,ki} + h_{kk,ij} - a_{,ij} + \dot{\beta}_{i,j} + \dot{\beta}_{j,i} = 0$$

$$\Longrightarrow \begin{cases} \ddot{h}_{ij} = \Delta h_{ij} \\ \ddot{\beta}_i = \Delta\beta_i \\ \ddot{a} = \Delta a + 2\underline{\left(h_{ik,ki} - h_{kk,ii}\right)} = \Delta a \end{cases}$$

Hamiltonian constraint, → 0

Hyperbolic equations are obtained

problems

- Problem I: Show $n^{\alpha}\nabla_{\alpha}n_{\mu} = D_{\mu}\ln\alpha$
- Problem II: derive the evolution equations of the Hamiltonian and momentum constraints
- Problem III: Derive the linear 3+1 equations for $\alpha \neq 1$ & $\beta^{k} \neq 0$
- Problem IV: Find a gauge condition which leads to $\ddot{h}_{ij} = \Delta h_{ij}$ in the linearized Einstein's equation