GENERALISED HARMONIC EVOLUTION SCHEME

Harmonic coords

Einstein 1912 during development of NR

De Donder 1921 "de Donder gauge"

Choquet-Bruhat 1952
Fischer + Marsden 1972 } well-posedness of GR

Fock 1955 analyse GWs

in NR:

Garfinkle 2002 singularities

Szilagyi et al 02, 03 linearized GR

Generalized Harmonic

Friedrich 1985 (hyperbolicity)

PRETORIUS 2005 first BBH merger

SpEC/SXS 2005 – now

Basic Idea

Derive evolution eqs directly from Einstein Eqs

$$G_{\mu\nu} = 8\pi\, T_{\mu\nu}$$

$$\Rightarrow \quad R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \right)$$

expand Ricci:

Wave operator

$\Gamma_\nu = g^{\rho\sigma} \Gamma_{\nu\rho\sigma}$

"non-hyperbolic" 2nd order terms

$$R_{\mu\nu} = -\frac{1}{2} \boxed{g^{\rho\sigma} \partial_\rho \partial_\sigma g_{\mu\nu}} + \nabla_{(\mu} \Gamma_{\nu)}$$

$$\underbrace{+ g^{\rho\sigma} g^{\lambda\vartheta} \left(\partial_\lambda g_{\rho\mu} \partial_\vartheta g_{\sigma\nu} - \Gamma_{\mu\rho\lambda} \Gamma_{\nu\sigma\vartheta} \right)}_{\text{lower order terms (1st derivs)}}$$

can show $\quad \Gamma_\mu = -g_{\mu\nu} \nabla_\rho \nabla^\rho x^\nu$

Harmonic coords: $\quad \nabla_\rho \nabla^\rho x^\nu = 0 \Rightarrow \Gamma_\mu = 0 \Rightarrow \nabla_{(\mu} \Gamma_{\nu)} = 0$

$\Rightarrow$ E. Eqs. reduce to wave-equations for each component of the metric $g_{\mu\nu}$

__Generalized__ Harm. Coords: $\quad g_{\mu\nu} \nabla_\rho \nabla^\rho x^\nu = H_\mu (t, x^i; g_{\mu\nu})$

$$\Rightarrow \quad \nabla_\mu \Gamma_\nu = -\nabla_\mu H_\nu (t, x^i; g_{\mu\nu}) \qquad \text{1st derivs of } g_{\mu\nu} \text{ only}$$

$\Rightarrow$ E. Eq. still wave-equations, with $\nabla_\mu H_\nu$ part of lower order terms

To Do

1) How do $\underset{\sim}{H}_\mu$ determine gauge? How to choose them?

2) new constraint $\underset{\sim}{C}_\mu = \underset{\sim}{H}_\mu + \underset{\sim}{\Gamma}_\mu$ must be conserved

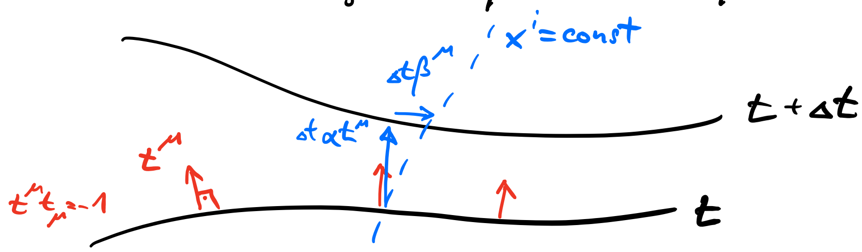
3) implementation: how to solve GH eqns?

4) Boundary conditions

1) Meaning of H_μ

perform 3+1 split

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$



can show

$$\partial_t \alpha - \beta^k \partial_k \alpha = -\alpha\left(H_t - \beta^i H_i + \alpha K\right)$$

$$\partial_t \beta^i - \beta^k \partial_k \beta^i = \alpha \gamma^{ij}\left[\alpha\left(H_j + g^{k\ell}\Gamma_{jk\ell}\right) - \partial_j \alpha\right]$$

$H_t, H_i \simeq$ time-derivatives of lapse + shift

2) GH constraint evolution

$$C_\mu := H_\mu - \Gamma_\mu \quad = 0$$ supposed to be

Einstein Eq. with $\nabla\Gamma \to \nabla H$ replacement

$$0 = R_{\mu\nu} - \nabla_{(\mu} C_{\nu)} \qquad (*)$$

Trace-reverse

$$0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \nabla_{(\mu} C_{\nu)} + \frac{1}{2} g_{\mu\nu} \nabla_\rho C^\rho$$

Divergence

$$0 = \underbrace{\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right)}_{=0 \text{ Bianchi}} - \nabla^\mu \left(\nabla_{(\mu} C_{\nu)} \right) + \frac{1}{2} g_{\mu\nu} \nabla^\mu \nabla_\rho C^\rho$$

$$\nabla_\mu \nabla_\nu C^\mu = \nabla_\nu \nabla_\mu C^\mu + R_{\mu\nu} C^\mu$$

$\times(-2)$

$$0 = \nabla^\mu \nabla_\mu C_\nu + R_{\mu\nu} C^\mu$$

$$\Big\| \quad 0 = \nabla^\mu \nabla_\mu C_\nu + C^\mu \nabla_{(\mu} C_{\nu)} \quad \Big\|$$

$C_\mu = 0$ analytically conserved:

if at $t=0$ $C_\mu = \dot{C}_\mu = 0 \Rightarrow C_\mu \equiv 0$

Relation to usual Hamiltonian & Momentum constraint

$$(H, M_i) = \mathcal{M}_\mu := G_{\mu\nu} t^\nu$$

unit-normal to $t =$ const foliation, $t^\nu t_\nu = -1$

$$= \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right) t^\nu$$

$0 = R_{\mu\nu} - \nabla_{(\mu} C_{\nu)}$

$$= \left(\nabla_{(\mu} C_{\nu)} - \tfrac{1}{2} g_{\mu\nu} \nabla_\rho C^\rho\right) t^\nu$$

$$t^\nu \nabla_\nu C_\mu = 2\mathcal{M}_\mu - t^\nu \nabla_\mu C_\nu + g_{\mu\nu} t^\nu \nabla_\rho C^\rho$$

$$\vdots$$

$$t^\nu \nabla_\nu C_\mu = 2\mathcal{M}_\mu + \left(\gamma^{\nu\rho} t_\mu - \gamma_\mu{}^\nu t^\rho\right) \nabla_\mu C_\rho$$

$\Rightarrow$ ADM constraints $\mathcal{M}_\mu = (H, M_i)$ are derivatives of GH constraints C_μ.

C_μ contain <u>1st derivs</u> only. Easier to do constraint-damping.

Constraint damping

GH as described so far leads to fast growth of (initially small) C_μ.

The fix (Pretorius '05, Gundlach et al '05)

$$0 = \underbrace{R_{\mu\nu} - \nabla_{(\mu} C_{\nu)}}_{\text{as before: } \Box g_{\mu\nu} = \text{l.o.}} + \gamma_0 \left(t_{(\mu} C_{\nu)} - \frac{1}{2} g_{\mu\nu} t^\rho C_\rho \right)$$

new term!

has no effect, if $C_\mu = 0$

same calc as above

$$\Rightarrow \quad 0 = \nabla^\mu \nabla_\mu C_\nu + C^\mu \nabla_{(\mu} C_{\nu)} - 2\gamma_0 \nabla^\mu \left(t_{(\mu} C_{\nu)} \right) - \frac{\gamma_0}{2} t_\nu C_\rho C^\rho$$

if $C_\mu \ll 1$

$$0 = \nabla^\mu \nabla_\mu C_\nu - 2\gamma_0 \nabla^\mu \left(t_{(\mu} C_{\nu)} \right)$$

in short-wavelength limit, can show that $C \sim e^{-\gamma_0 t}$ or $e^{-\gamma_0 t/2}$

small C-violations damped away!

3) Implementing GH as first order system

Goal $\partial_t \underline{u} + \underline{\underline{A}}^k \partial_k \underline{u} = \underline{F}$

Have $\Box g_{\mu\nu} = $ lower order

$\underline{u} = \{ g_{\mu\nu}, \Pi_{\mu\nu}, \phi_{i\mu\nu} \}$

$\Pi_{\mu\nu} \sim -\dot{g}_{\mu\nu}$, $\quad \phi_{i\mu\nu} = \partial_i g_{\mu\nu}$

Def of $\Pi_{\mu\nu}$

$$\Rightarrow \quad \partial_t g_{\mu\nu} - \beta^k \partial_k g_{\mu\nu} = -\alpha \Pi_{\mu\nu} + \gamma_1 \beta^i C_{i\mu\nu}$$

$$\partial_t \phi_{i\mu\nu} - \beta^k \partial_k \phi_{i\mu\nu} + \alpha \partial_i \Pi_{\mu\nu} = \text{l.o.} + \gamma_2 C_{i\mu\nu}$$

$$\partial_t \Pi_{\mu\nu} - \beta^k \partial_k \Pi_{\mu\nu} + \alpha \gamma^{ki} \partial_k \phi_{i\mu\nu} = \text{l.o.} + \gamma_3 \beta^i C_{i\mu\nu}$$

$\partial_t \Pi_{\mu\nu} \sim -\ddot{g}_{\mu\nu}$, $\quad \alpha \gamma^{ki} \partial_k \phi_{i\mu\nu} \sim \nabla^2 g_{\mu\nu}$, $\quad$ l.o.: GH rhs

Add multiples of first order reduction constraint

$$C_{i\mu\nu} := \partial_i g_{\mu\nu} - \phi_{i\mu\nu}$$

$\gamma_2 > 0$: damps $C_{i\mu\nu} \to 0$

$\gamma_1 = -1$: achieves linear degeneracy (no shocks)

$\gamma_3 = \gamma_1 \gamma_2$: symmetric hyperbolicity

4) Boundary conditions

$$\partial_t \underline{u} + \underline{\underline{A}}^k \partial_k \underline{u} = \ell.o.$$

eigen value problem determines characteristic fields $\underline{e}^A$

$$\underline{e}^A \left(\underline{\underline{A}}^k n_k \right) = v_{(A)} \, \underline{e}^A$$

most important eigenvector

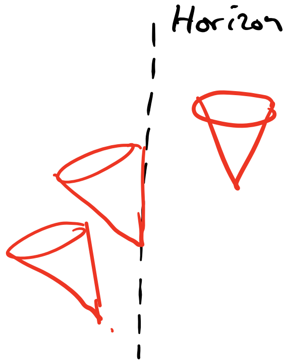
$$u^{1\pm}_{\mu\nu} = \Pi_{\mu\nu} \pm n^k \phi_{k\mu\nu} - \gamma_2 g_{\mu\nu} \qquad v_1^\pm = \pm \alpha - n_k \beta^k$$

$$\sim -\dot{\psi} \pm \partial_n \psi \qquad\qquad \sim \pm 1$$

mode travelling with $v = c$ in direction $\pm \hat{n}^k$

all other modes inside light cone

a) inside BH Horizons



inside Horizon: all modes ingoing

no BC needed.

b) at large distance (outer bdry)

$u^{1-}_{\mu\nu}$ has 10 components that must be specified

4 dof determined by constraints vanishing

2 dof physical BC: incoming GW h_+, $h_\times$

4 dof determine gauge (i.e. outer BC on $\Box x^\mu = H^\mu$)