IMPRS GW Astronomy – Computational Physics 2022

Problem Set 2, Part I: ODEs.

Serguei Ossokine, Vijay Varma

1 RK methods

We saw in the lectures that for a given order, the RK approach produces a family of methods. Here we explore some aspects of this.

Task RK-1:

Derive the most general explicit second-order RK method. Show in particular that its Butcher tableau is given by

$$\begin{array}{c|cc} 0 & & \\ \alpha & \alpha & \\ \hline & \left(1 - \frac{1}{2\alpha}\right) & \frac{1}{2\alpha} \end{array} \tag{1}$$

Task RK-2:

Make your own. Implement 2 new different 2nd order RK methods and evolve the same problem as given in the accompanying notebook. How do their accuracy compare to the classical second-order methods (make a plot of absolute error with respect to the analytic solution)?

Task RK-3:

In class we saw the region of stability for Euler's method. Write a program to draw the region of stability for Forward Euler, Midpoint and classical RK4 methods.

Task RK-4:

The total error (consisting of roundoff and truncation errors) for a given nth-order method can be schematically written as $E \sim \epsilon/h + h^n$, where ϵ is the machine epsilon ($\simeq 2.22 \times 10^{-16}$ for 64 bit double-precision). Estimate the "break-even" step size h for where the total error is minimized (i.e. find the local minimum of the error) and plot the result as a function of n.

2 Kepler problem

Next, let's use a Scipy ODE solver with adaptive time stepping to solve the Kepler problem for a comet in an elliptical orbit around the Sun, as shown in Fig. 1.

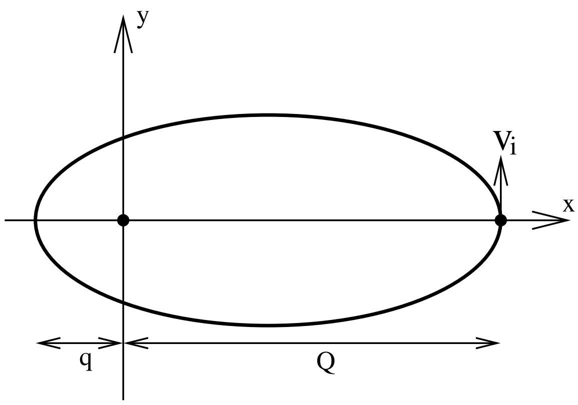

Figure 1: Trajectory of a comet orbiting the Sun.

Here, the Sun is at origin (the focus of the ellipse), the semimajor axis is along the x-axis, and the semiminor axis is along the $y-$axis. q denotes the distance to the perihelion (point of closest approach, while Q denotes the distance to the aphelion (point of furthest approach). The equations of motion are given by:

$$\ddot{\boldsymbol{r}} = -\frac{GM}{r^2}\,\hat{\boldsymbol{r}} \tag{2}$$

Where M is the mass of the Sun. Being good physicists, we will work with units of $G = 1$. To simplify notation and code, we will also use $M = Q = 1$. Then, the comet's orbit in Fig. 1 has an eccentricity e, and is initialized at the aphelion $\boldsymbol{r}_0 = (1, 0)$, with initial velocity $\dot{\boldsymbol{r}}_0 = (0, V_i)$, with $V_i = \sqrt{1-e}$. The period of the orbit is given by $T = \frac{2\pi}{(1+e)^{3/2}}$. See https://young.physics.ucsc.edu/115/kepler.pdf for a derivation of these relations. Working in Cartesian coordinates, with $r = \sqrt{x^2 + y^2}$, the ODE problem to solve is:

$$\begin{aligned} \ddot{\boldsymbol{x}} &= -\frac{1}{r^3}\,\boldsymbol{x}, & \ddot{\boldsymbol{y}} &= -\frac{1}{r^3}\,\boldsymbol{y}, \\ \boldsymbol{x}_0 &= 1, & \boldsymbol{y}_0 &= 0, \\ \dot{\boldsymbol{x}}_0 &= 0, & \dot{\boldsymbol{y}}_0 &= \sqrt{1-e}, \end{aligned} \tag{3}$$

over the time interval $t = [0, T]$.

Task KP-1:

The ODEs in Eq. (3) are of second-order. Reduce them to first order using auxiliary variables. For your choice of $e < 1$, solve the first-order ODE equations using `scipy.integrate.solve_ivp`, with `method='RK45'`, and `atol=rtol=1e-7`. Produce a plot similar to Fig. 1.

Task KP-2:

The exact solution is given by

$$r = \frac{1-e}{1-e\,\cos\theta}, \tag{4}$$

where $\theta = \arctan\left(y/x\right)$.
Evaluate the ODE solution on a time array with 1000 uniform steps between $[0,T]$. Compute $r(t)$ and $\theta(t)$ and use them to compute:

$$err_r(t) = \left| r(t) - \frac{1-e}{1-e\,\cos\theta(t)} \right|, \tag{5}$$

Plot $err_r(t)$. Is $err_r(t)$ comparable to the requested tolerance?

Task KP-3:

Repeat Tasks KP-1 and KP-2 for $ecc = \{0, 0.1, 0.5, 0.9, 0.99\}$. Notice how the adaptive time stepper takes small steps near the perihelion and big steps near the aphelion, especially as you increase ecc.