# Solving Constraints

3-D $R[\gamma_{ij}]$

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho \qquad (H) \qquad \text{Hamiltonian}$$

$$D_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi S^i \qquad (M) \qquad \text{Momentum}$$

$$K = \gamma^{ij}K_{ij}$$

strategy: use potentials to isolate fixed degrees of freedom

- Conformal trafo

$$\gamma_{ij}(x^i) = \psi^4(x^i)\,\tilde{\gamma}_{ij}(x^i) \qquad \psi(x) > 0$$

conformal metric w/ its conformal diff. geo.

$$\tilde{\gamma}^{ik}\tilde{\gamma}_{kj} = \delta^i{}_j$$

$$\tilde{\Gamma}^i{}_{jk},\ \tilde{D}_i,\ \tilde{R}^l{}_{ijk},\ \tilde{R}_{ij},\ \tilde{R}$$

plug $\psi^4\tilde{\gamma}_{ij}$ into formulae

$$\Rightarrow \quad R = \psi^{-4}\tilde{R} - 8\psi^{-5}\tilde{D}^2\psi$$

into $H$ $\Rightarrow$

$$\boxed{\tilde{D}^2\psi - \frac{1}{8}\psi\tilde{R} + \frac{\psi^5}{8}\left(K_{ij}K^{ij} - K^2\right) = -2\pi\psi^5\rho \qquad (H')}$$

given $\tilde{\gamma}_{ij}$, this is elliptic eqn for $\psi$.

Aside: assume

$$\left.\begin{array}{l}\text{vacuum}\\ K_{ij}=0\end{array}\right\} \Longrightarrow \begin{array}{l}(M)\ \text{solved "}0=0\text{"}\\ (H') \Rightarrow \tilde{D}^2\psi = \frac{1}{8}\psi\tilde{R}\end{array}$$

also conformal flatness

$$\tilde{\gamma}_{ij} = f_{ij} = \begin{cases}\delta_{ij} & x, y, z\\ \mathrm{diag}(1, r^2, r^2\sin^2\theta) & r, \theta, \varphi\end{cases}$$

flat metric

concrete choice of coords

$$H' \Rightarrow \nabla^2_{\text{flat}}\psi = 0$$

Solved, e.g., by

$$\psi = \frac{M}{2r} + 1 \qquad M = \text{const}$$

$$ds^2 = \left(\frac{M}{2r}+1\right)^4 \left(dr^2 + r^2 d\Omega^2\right)$$

Schwarzschild in isotropic coords

or

$$\psi = \frac{M_1}{2|\vec{x}-\vec{c}_1|} + \frac{M_2}{2|\vec{x}-\vec{c}_2|} + 1$$

two BHs at $\vec{c}_1, \vec{c}_2$ (nonspinning, at rest)

- <u>BH linear momentum + spin</u>

Transverse-traceless decomp of $K_{ij}$

$$\underset{6\,\mathrm{DoF}}{K^{ij}} = \underbrace{A^{ij}}_{5,\ \text{trace free}} + \frac{1}{3}\gamma^{ij} \underset{1}{K}$$

$$A^{ij} = \psi^{-10} \tilde{A}^{ij}$$

$$\tilde{A}^{ij} = \underset{3\,\mathrm{DoF}}{(\tilde{\mathbb{L}}V)^{ij}} + \underset{2\,\mathrm{DoF}}{\tilde{A}^{ij}_{TT}}$$

$$(\tilde{\mathbb{L}}V)^{ij} = \tilde{D}^i V^j + \tilde{D}^j V^i - \frac{2}{3}\tilde{\gamma}^{ij} \tilde{D}_k V^k$$

$$\tilde{D}_i \tilde{A}^{ij}_{TT} = 0 = \tilde{\gamma}_{ij} \tilde{A}^{ij}_{TT}$$

transverse, traceless

Reason:

$$D_i A^{ij} = D_i\left(\psi^{-10}\tilde{A}^{ij}\right)$$

conformal identity $\forall$ trace-free, symmetric $A^{ij}$

$$= \psi^{-10} \tilde{D}_i \tilde{A}^{ij}$$

$$= \psi^{-10} \tilde{D}_i \left((\tilde{\mathbb{L}}V)^{ij} + \tilde{A}^{ij}_{TT}\right)$$

$$= \psi^{-10} \underbrace{\tilde{D}_i (\mathbb{L}V)^{ij}}_{\text{elliptic operator acting on } V^i \text{ good!}}$$

$$\overset{(M)}{\Rightarrow} \quad \tilde{D}_j (\tilde{\mathbb{L}}V)^{ij} - \frac{2}{3}\psi^6 \tilde{\gamma}^{ij} \partial_j K = 8\pi \psi^{10} S^i \qquad (M'')$$

$$\overset{(H)}{\Rightarrow} \quad \tilde{D}^2 \psi - \frac{1}{8}\psi \tilde{R} - \frac{\psi^2}{8} K^2 + \frac{\psi^{-7}}{8}\left(\tilde{\mathbb{L}}V + \tilde{A}_{TT}\right)\left(\tilde{\mathbb{L}}V + \tilde{A}_{TT}\right) = -2\pi \psi^5 \rho \qquad (H'')$$

Overall strategy

- choose $\tilde{\gamma}_{ij}$, $K$, $\tilde{A}^{ij}_{TT}$
- choose BCs to enforce BHs
- solve $H''$, $M''$ for $\psi$, $V^i$ four coupled elliptic PDE
- assemble $\gamma_{ij}$, $K^{ij}$

Notes:

- often simplified

$$\left.\begin{aligned} \tilde{\gamma}_{ij} &= f_{ij} \\ K &= 0 \\ \tilde{A}^{ij}_{TT} &= 0 \end{aligned}\right\} \Rightarrow \text{analytic "Bowen-York" solutions for } (M'').$$

$\Rightarrow$ analytic "Bowen-York" solutions

$$\tilde{A}^{ij}_{P} = \frac{3}{2r^2}\left(P^i n^j + P^j n^i - (\delta^{ij} - n^i n^j)\, P^k n_k\right)$$

$$r = |\vec{x} - \vec{c}_{BH}| \qquad n^k = \frac{\vec{x} - \vec{c}_{BH}}{r}$$

$$\tilde{A}^{ij}_{S} = \frac{6}{r^3}\; n^{(i}\,\epsilon^{j)kl}\, n_k\, S_l$$

multi-BH:
$$\tilde{A}^{ij} = \sum_{\alpha=1}^{N_{BH}} \tilde{A}^{ij}_{P_\alpha} + \tilde{A}^{ij}_{S_\alpha}$$

total spacetime momentum

$$P^i_{ADM} = \frac{1}{8\pi}\oint_\infty \left(K^{ik} - \gamma^{ik} K\right) dA_k \overset{\psi\to 1,\ r\to\infty}{=} \dots = \sum_\alpha P^i_\alpha$$

$P^i_\alpha$ = linear momentum of BH $\alpha$

$$J^i_{ADM} = \dots = \sum_\alpha S^i_\alpha + \varepsilon^i{}_{k\ell}\, c^k_{BH,\alpha}\, P^\ell_\alpha$$

$S^i_\alpha$ = angular momentum of BH $\alpha$

Know $P_{ADM}$, $J_{ADM}$ <u>before</u> solving (H).

- to enforce BH, need $\psi \to \infty$ at $x_{BH}$. Write

$$\psi(\vec{x}) = u(\vec{x}) + \underbrace{\sum_\alpha \frac{m_\alpha}{2r_\alpha}}_{=: \frac{1}{a}}$$

$a = O(r_\alpha)$ near $\vec{c}_\alpha$

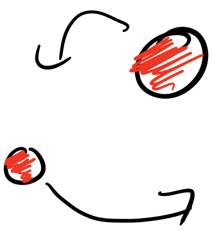
$$\Rightarrow \nabla^2 u + \underbrace{\frac{1}{8} a^7 \underbrace{\tilde{A}^{ij}_{BY}}_{\sim r_\alpha^{-3}} \tilde{A}^{BY}_{ij}}_{O(r_\alpha)} \underbrace{(1+ua)}_{\text{finite}}{}^{-7} = 0$$

PUNCTURE DATA (Brandt + Brügmann 1998)

$\Rightarrow$ just solve on $\mathbb{R}^3$

\+ easy to implement — only BH spin $\lesssim 0.90$

# Conformal thin sandwich ID



2 BH's in circular orbit

in co-rotating frame $\partial_t \approx 0$

want to specify

$$\tilde\gamma_{ij}\,, \quad \partial_t \tilde\gamma_{ij} = 0$$

def. $K_{ij} \approx \partial_t \gamma_{ij} + D_i \beta_j + D_j \beta_i$

$$\Rightarrow K^{ij} = \psi^{-10} \frac{1}{2\tilde\alpha} (\tilde{\mathbb{L}}\beta)^{ij} + \gamma^{ij} K$$

shift

$\tilde\alpha = \psi^{-6}\alpha$ lapse

$$= \frac{1}{2\tilde\alpha}(\tilde{\mathbb{L}}\beta)^{ij}$$

in (M)

$$\tilde\nabla_i \frac{1}{2\tilde\alpha}(\tilde{\mathbb{L}}\beta)^{ij} - \frac{2}{3}\psi^6 \tilde\nabla^i K = 0$$

$$(H) \Rightarrow \quad \tilde\nabla^2 \psi - \frac{1}{8}\psi \tilde R - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^{-7}\tilde A_{ij}\tilde A^{ij} = 0$$

Furthermore, from ADM eqn for $K_{ij}$

$$\partial_t K_{ij} = \dots \quad \Rightarrow \quad \partial_t K = -\nabla^2 \alpha + \dots$$

$$\partial_t K = 0 \Rightarrow \quad \tilde\nabla^2(\psi^7\tilde\alpha) - (\psi^7\tilde\alpha)\left[\;\dots\;\right] - \psi^5 \beta^k \partial_k K = 0$$

"XCTS" equations

$\tilde{\gamma}_{ij}$, $\partial_t \tilde{\gamma}_{ij} = 0$ ; $K$, $\partial_t K = 0$

$\Rightarrow$ 5 PDEs for $\psi$, $\beta^i$, $\alpha$

- lapse + shift in 1D ?

If one evolves with $\alpha$, $\beta^i$ then

$$\partial_t \tilde{\gamma}_{ij} = 0$$

$$\partial_t K = 0$$

- more flexibility than puncture data

any $\tilde{\gamma}_{ij}$, $K$

specifically $\tilde{\gamma}_{ij} = \sum \gamma_{ij}^{Kerr}$

allows BHs spins near 1