Hyperbolic Eqns

Discretisation similar to elliptic eqns, but no linear system to solve

Eg $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$

Finite differences

$$u_j = u(x_j) \qquad x_j = hj, \quad h = \frac{1}{N}, \quad j = 0, \dots, N-1$$

$$\dot{u}_j = \frac{u_{j+1} - u_{j-1}}{2h} + O(h^2)$$

$$\text{or} \quad \frac{-u_{j+2} + 8u_{j+1} - 8u_{j-1} + u_{j-2}}{12h} + O(h^4) \qquad O(h^{2k}) \leftrightarrow \text{stencil with } 2k+1$$

Set of ODEs for grid-point values $\{u_j\} = \underline{u}$

$$\underline{\dot{u}} = \underline{F}[\underline{u}]$$

Solve with your favourite timestepper
Most simplistically

$$u(jh, k\Delta t) = u_j^{(k)}$$

$$\dot{u}_j^{(k)} = \frac{u_j^{(k+1)} - u_j^{(k)}}{\Delta t} + O(\Delta t)$$

$$\Rightarrow u_j^{(k+1)} = u_j^{(k)} + \Delta t \frac{u_{j+1} - u_{j-1}}{2\Delta x} + O(\Delta t^2) \qquad \text{"Forward-Euler"}$$

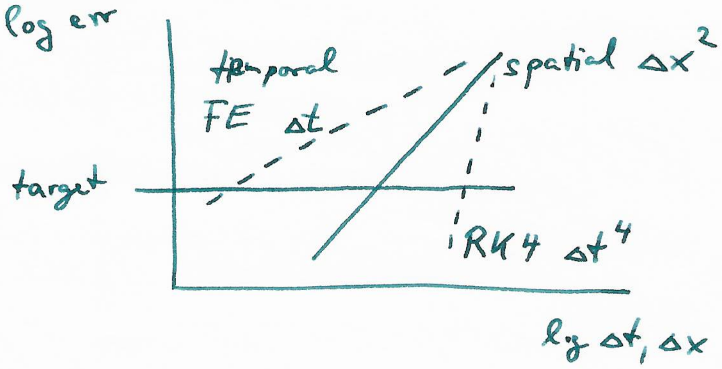
To evolve to T need $\sim \frac{1}{\Delta t}$ steps.

Error

$$O\left(\Delta t^2 \cdot \frac{1}{\Delta t}\right) = O(\Delta t) \qquad \text{first order conv.}$$

Usually use tstepper w/ faster convergence, Runge-Kutta 4 Δt^4

Balance accuracy



FE $\Rightarrow$ requires excessively small Δt

RK4 + $O(\Delta x^2)$: large $\Delta t \sim \Delta x^{1/2}$ seems optimal

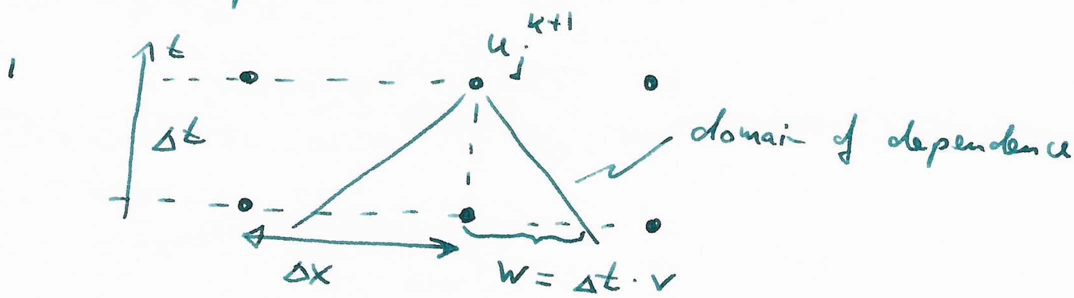
However, COURANT LIMIT

stability $\Leftrightarrow$ $\dfrac{\Delta t}{\Delta x} \leq C \, (\max v)^{-1} \sim C$

largest char speed. $\qquad v \sim c = 1$

$C = O(1)$ depends on tstepper, discretisation.

Heuristic explanation



if $w > \Delta x$, u_j^{k+1} requires info from outside stencil ↯

FD summary

+ easy
+ robust
- accuracy
- wide stencils need lots of communication

Spectral Methods

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

$$u_i = \sum_{k=0}^{N-1} \tilde{u}_k \, \phi_k(x_i)$$

$$\frac{\partial u_i}{\partial x} = \left(A \tilde{D} A^{-1}\right)_{ij} u_j$$

$$\dot{\underline{u}} = \left(\underline{A} \, \underline{\tilde{D}} \, \underline{A}^{-1}\right) \underline{u}$$

$\Longrightarrow$ method of lines

often RK4 – requires care to balance resolution via N, Δt

$\hookrightarrow$ Problem Set 1 !

SpEC uses Dormand Prince 853 $O(\Delta t^8)$ w/ Δt-control

+ very accurate when applicable

+ sensitive to mathematically correct formulation of PDE + BC

– not for shocks

– limited parallelisability

Why exponential convergence?

Fourier Series

$$u(x) = \sum \tilde{u}_k \, e^{ikx} \qquad x \in [-\pi, \pi]$$

$$2\pi \, \tilde{u}_k = \int_{-\pi}^{\pi} u(x) \, e^{ikx} \, dx$$

$$= \underbrace{\frac{i}{k} \, u(x) \, e^{ikx} \Big|_{-\pi}^{\pi}}_{\substack{(-1)^k (u(\pi) - u(-\pi)) \\ =0 \text{ if periodic}}} + \left(\frac{-i}{k}\right) \int_{-\pi}^{\pi} u'(x) \, e^{-ikx} \, dx$$

$$= \frac{-i}{k} \frac{i}{k} \underbrace{u'(x) \, e^{ikx} \Big|_{-\pi}^{\pi}}_{=0 \text{ if periodic}} + \left(\frac{-i}{k}\right)^2 \int_{-\pi}^{\pi} u''(x) \, e^{-ikx} \, dx$$

$$\vdots$$

$$+ \left(\frac{-i}{k}\right)^n \int_{-\pi}^{\pi} u^{(n)}(x) \, e^{-ikx} \, dx$$

if u periodic & C^∞, after n integr. by parts

$$\tilde{u}_k \sim \frac{1}{k^n}$$

$\forall n \Rightarrow \tilde{u}_k$ decay exponentially

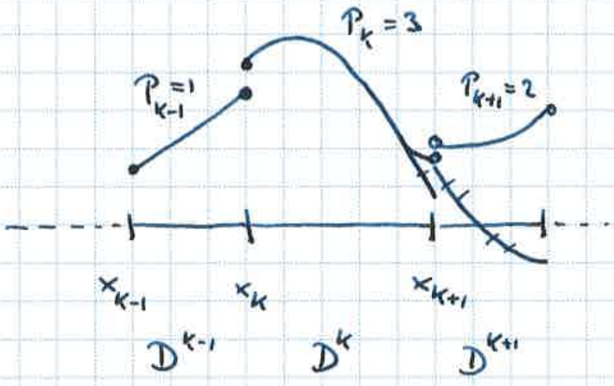
Chebyshev
$T_k(x) = \cos(k t)$, Fourier series in $t = \arccos x$

Legendre: Integration by parts works b/c singular Sturm-Liouville problem

$\Rightarrow$ boundary term vanishes

DISCONTINUOUS GALERKIN

Approx: many cells, each w/ ~~separate basis~~ spectral expansion



$$x \in D^k: \quad u_k^{(P_k)}(x,t) = \sum_{n=0}^{P_k} \tilde{u}_n^k(t)\, \phi_n^k(x)$$

span polynomials (Legendre poly)

$\{u_k^{P_k}\}$

Eqns: Residual $R(u_h) = \dot{u}_h - u_h'$

orthogonal to basis-functions

$$\int_D R(u_h)\, \phi_{n_0}^{k_0}(x)\, dx = 0 \quad \forall k_0, n_0$$

Integr. by parts

$$= \sum_{D^k} \dot{\tilde{u}}_n^k \underbrace{\int_{D^k} \phi_n^k(x)\, \phi_{n_0}^{k_0}(x)\, dx}_{} + \tilde{u}_n^k \underbrace{\int_{D^k} \phi_n^k(x) \frac{d\phi_{n_0}^{k_0}(x)}{dx}\, dx}_{} - \int_{\partial D^k} \tilde{u}_n^k \phi_n^k \phi_{n_0}^{k_0}\, d\vec{A}$$

$$\underline{\underline{M}} \{\dot{\tilde{u}}_n^k\} + \underline{\underline{S}} \{\tilde{u}_n^k\} + \text{Flux}$$

mass-matrix — stiffness matrix — contains interdomain matching

$$\{\dot{\tilde{u}}_n^k\} + \underline{\underline{M}}^{-1} \underline{\underline{S}} \{\tilde{u}_n^k\} + \underline{\underline{M}}^{-1}\, \text{Flux}$$

once again, method of lines w/simple matrix operations

$\underline{\underline{M}}$, $\underline{\underline{S}}$ local to each D^k
Flux needs only neighbor bdry data

$\Bigg\}$ ⇒ good parallelisable

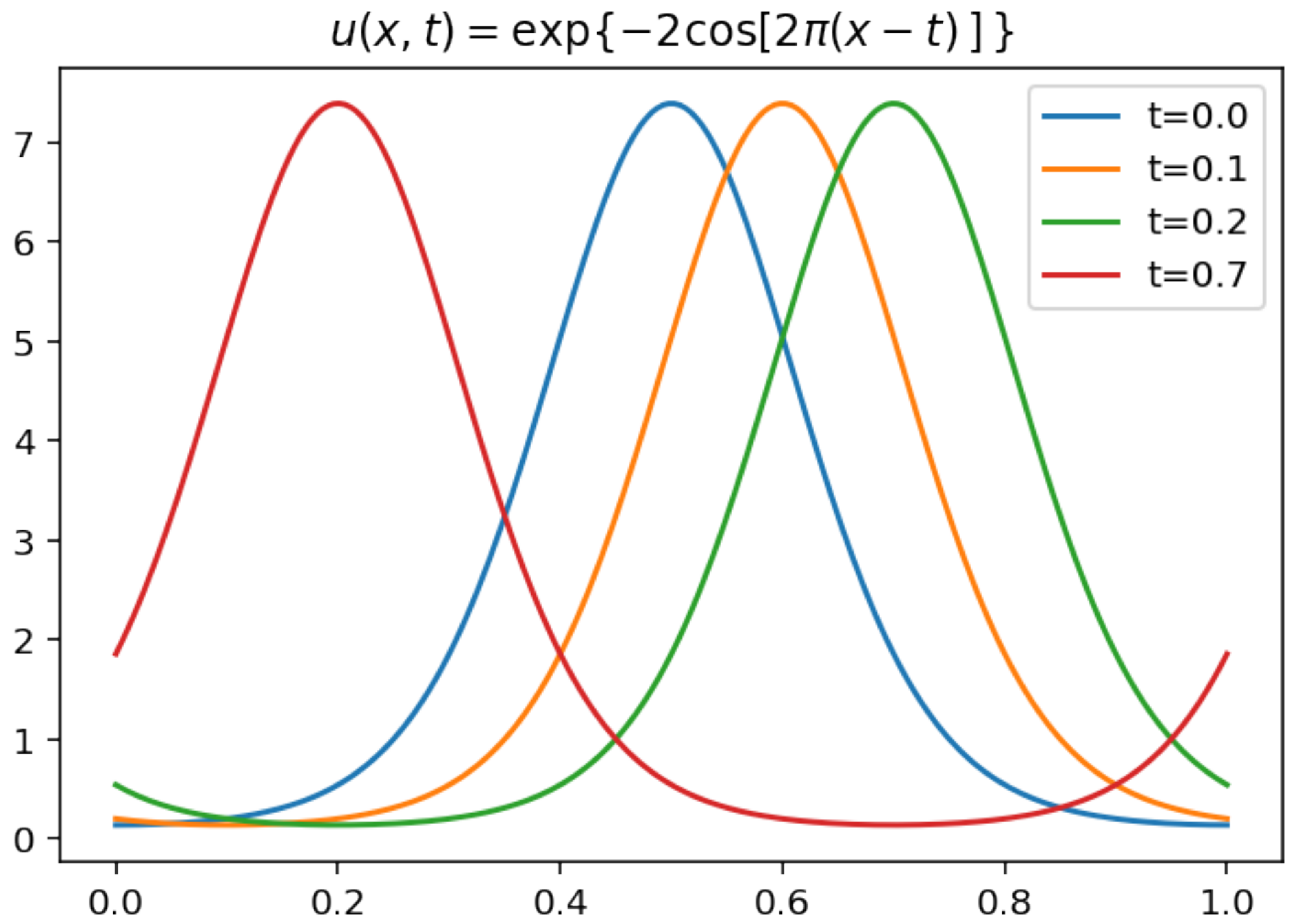
Accuracy control

P_k ~~[illegible]~~ ← when smooth, exp. conv.
~~size~~ divide/join D^k's ← near shocks

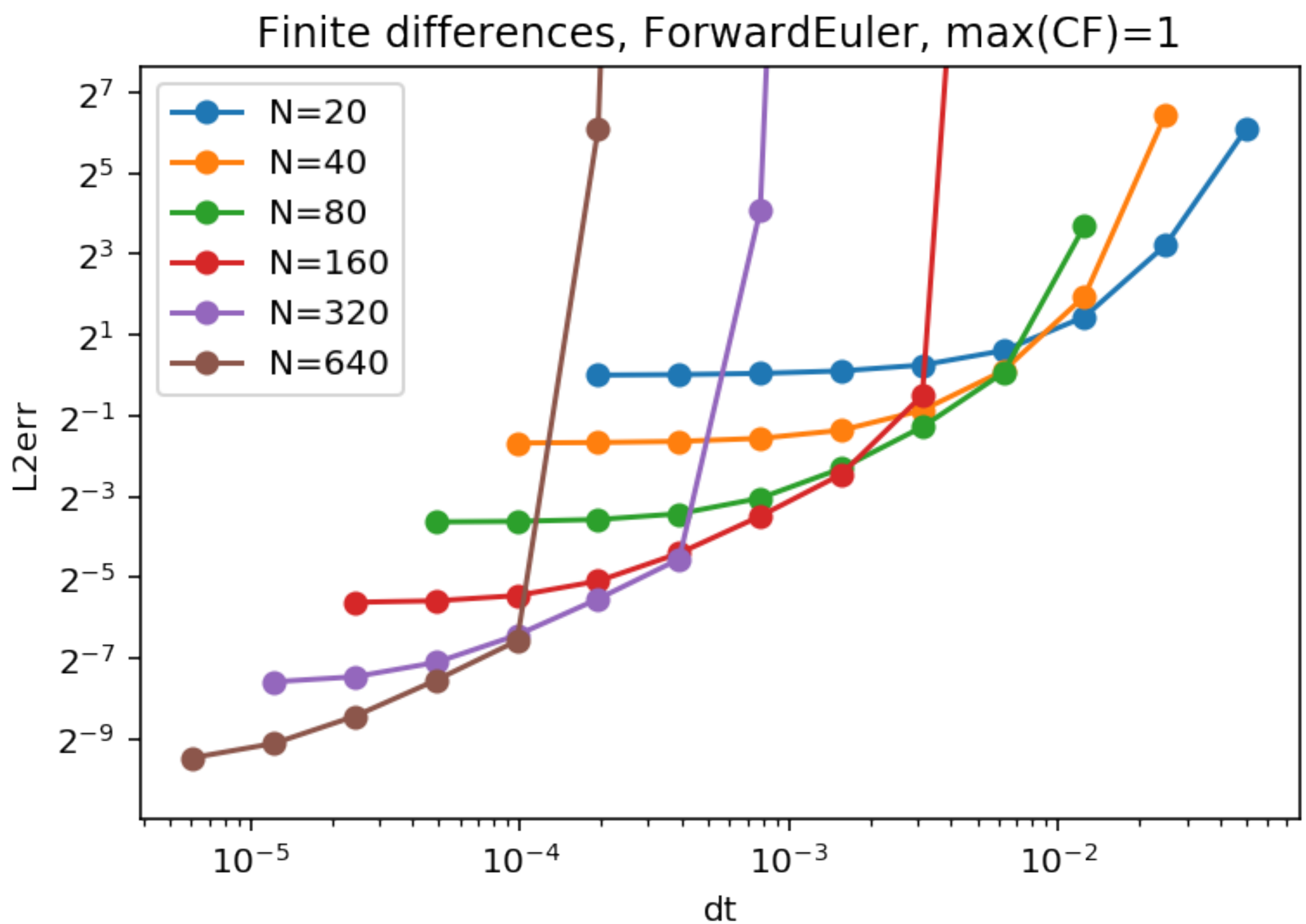
$\Bigg\}$ ⇒ flexible

Example solution, advection eqn

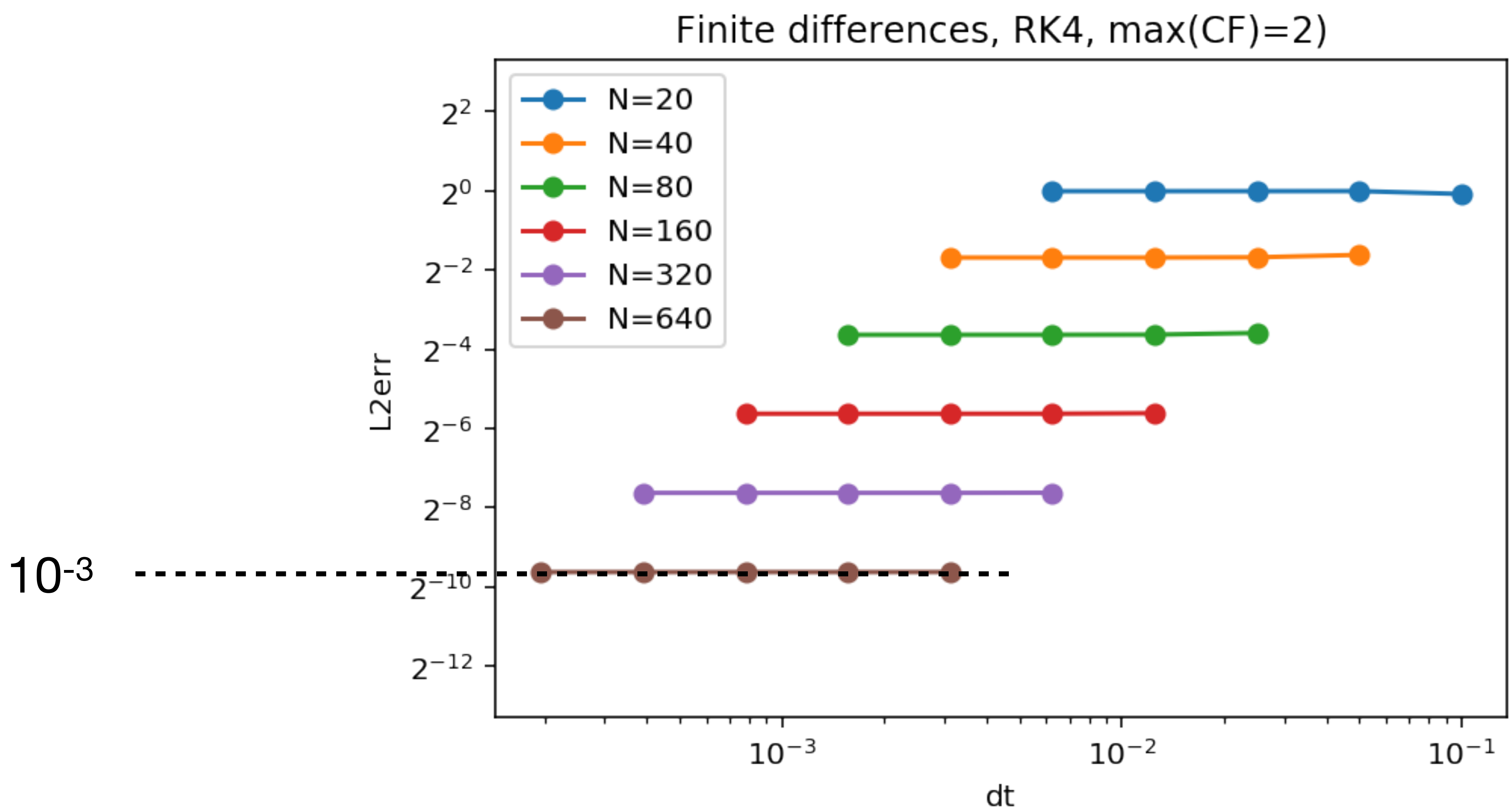
$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$



Convergence test



Convergence test 2



Convergence test 3