IMPRS GW Astronomy – Computational Physics 2025
BSSN evolution equations

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Notations used in this lecture note.

- $a, b, c....$ - spacetime indices
- $i, j, k....$ - spatial indices
- ${}^{(4)}R_{ab}$ - Ricci tensor
- g_{ab} - spacetime metric
- ${}^{(4)}R = {}^{(4)}R_{ab}g^{ab}$ - Ricci scalar
- γ_{ij} - spatial metric
- ${}^{(3)}R_{ij}$ - Ricci tensor associated with γ_{ij}
- ${}^{(3)}R = {}^{(3)}R_{ij}\gamma^{ij}$ - Ricci scalar
- D_i - spatial covariant derivative
- ∂_i - partial derivative
- $\Delta = \partial^i\partial_i$ - Laplacian
- ∇_a - spacetime covariant derivative
- Σ_t - hypersurface at t=const.
- n^a - future pointing vector normal to Σ_t
- K_{ij} - extrinsic curvature
- $K = K^i_i$
- α - Lapse function
- β^i - shift vector
- T_{ab} - stress-energy tensor
- $\Box = -\partial_t^2 + \Delta$ - d'Alembertian

1 What makes the original 3+1 Einstein system numerically unstable?

1.1 Recap of the standard 3+1 formalism

The $3+1$ formalism is expressed by two constraint equations

$$\begin{aligned} {}^{(3)}R - K_{ij}K^{ij} + K^2 &= 16\pi T_{ab}n^a n^b & (1) \\ D_i K^i_j - D_i K &= -8\pi T_{ab} n^a \gamma^b_j, & (2) \\ & & (3) \end{aligned}$$

and two evolution equations

$$\begin{aligned} \left(\partial_t - \mathcal{L}_\beta\right) K_{ij} &= -D_i D_j \alpha + \alpha \left({}^{(3)}R_{ij} + K K_{ij} - 2K_{il}K^l_j\right) \\ &\quad -8\pi\alpha T_{ab}\left[\gamma^a_i\gamma^b_j - \frac{1}{2}\left(\gamma^{ab} - n^a n^b\right)\gamma_{ij}\right] & (4) \\ \left(\partial_t - \mathcal{L}_\beta\right)\gamma_{ij} &= -2\alpha K_{ij}. & (5) \end{aligned}$$

1.2 Linearized evolution equations in vacuum

Below we discuss the behavior of gravitational wave propagation in vacuum spacetime ($T_{\alpha\beta} = 0$) in the linearized 3+1 formulation and find which term would cause the numerical difficulty. For simplicity we assume $\alpha = 1$ and $\beta^i = 0$. Neglecting higher order terms such as K^2, the evolution equations (4) and (5) are rewritten as

$$\begin{aligned} \partial_t K_{ij} &= {}^{(3)}R_{ij} & (6) \\ \partial_t \gamma_{ij} &= -2K_{ij}. & (7) \end{aligned}$$

By combining these two, we obtain

$$\partial_t^2 \gamma_{ij} = -2\ {}^{(3)}R_{ij}. \qquad (8)$$

As the Ricci tensor can be approximated by

$${}^{(3)}R_{ij} \approx \frac{1}{2}\left[-\Delta\gamma_{ij} + \gamma^{kl}\left(\partial_{lj}\gamma_{ik} + \partial_{li}\gamma_{jk} - \partial_{ij}\gamma_{kl}\right)\right], \qquad (9)$$

plugging ${}^{(3)}R_{ij}$ into Eq. (8) yields

$$\begin{aligned} \partial_t^2\gamma_{ij} &= \Delta\gamma_{ij} \\ &\quad -\gamma^{kl}\left(\partial_{lj}\gamma_{ik} + \partial_{li}\gamma_{jk} - \partial_{ij}\gamma_{kl}\right). & (10) \end{aligned}$$

The first line looks like a normal hyperbolic equation system, while we find an extra term in the second line. As we will have a look, this second line is the source of headache.

1.3 Stability check of 3+1 formalism

In the linearized system, where $\gamma_{ij} = \delta_{ij} + h_{ij}$ with $|h_{ij}| \ll 1$, Eq. (10) can be further rewritten as

$$\partial_t^2 h_{ij} = \Delta h_{ij} - (\partial_{kj} h_{ik} + \partial_{ki} h_{jk} - \partial_{ij} h_{kk}) . \tag{11}$$

A general solution to this equation is

$$h_{ij} = A\delta_{ij} + \partial_i B_j + \partial_j B_i + \partial_{ij} C + h_{ij}^{\rm TT}, \tag{12}$$

with A and C being a scalar, B_i being a divergence free vector (i.e. $\partial^i B_i = 0$), and $h_{ij}^{\rm TT}$ being a transverse trace-free (TT) tensor. After inserting the general solution 12 into both sides of Eq. (11), the propagation equation reads as

$$\begin{aligned} \partial_t^2 h_{ij} &= \ddot{A}\delta_{ij} + \partial_i \ddot{B}_j + \partial_j \ddot{B}_i + \partial_{ij} \ddot{C} + \ddot{h}_{ij}^{\rm TT} \\ &= \Delta A \delta_{ij} \qquad\qquad\qquad\qquad + \partial_{ij} A + \Delta h_{ij}^{\rm TT}. \end{aligned} \tag{13}$$

Therefore, by comparing the first and second line, we finally get the following evolution equations for each of newly introduced perturbed quantities

$$\ddot{A} = \Delta A \tag{14}$$

$$\ddot{h}_{ij}^{\rm TT} = \Delta h_{ij}^{\rm TT} \tag{15}$$

$$\partial_i \ddot{B}_j + \partial_j \ddot{B}_i = 0 \tag{16}$$

$$\ddot{C} = A. \tag{17}$$

In the above equations, the first two hyperbolic equations behave quite well, i.e. numerically stable. Regarding the third line, though I skip a detailed discussion about its behavior, it is known that it behaves also well, but as long as the momentum constraint is satisfied. The numerical instability of the original 3+1 formalism is coming from the last equation.

Now let's understand how badly it evolves. The Hamiltonian constraint in the linear approximation is written as

$$0 = \mathcal{H} = {}^{(3)}R - K_{ij}K^{ij} + K^2 \sim \Delta h - \partial^i \partial^j h_{ij}, \tag{18}$$

where $h \equiv h_{ii}$. Then from Eq. (11), we can derive $\Delta A = 0$, which leads to the Hamiltonian constraint $A = 0$ in asymptotically flat spacetime. In any numerical simulations it is inevitable to prevent a finite value of A, i.e., violation of the local Hamiltonian constraint. However, this local violation can be dispersed away via the first equation $\Box A = 0$, i.e. a wave equation, whose general solution in spherical symmetry is

$$A \propto \frac{e^{i(kr \pm \omega t)}}{r}. \tag{19}$$

Therefore the violation of local Hamiltonian constraint itself is not the major origin of numerical instability. Instead C appearing in the last equation (17) is the origin of problems, as explained below. We now have a look how C evolves. The general solution becomes

$$C = C_0 A + C_1 t + C_2, \tag{20}$$

where $C_{0,1,2}$ are coefficients. At a far distant region ($r \to \infty$), $A(\propto r^{-1})$ aproaches zero and the leading term is the second one, which may secularly increase unless C_1 is exactly zero. Consequently, h_{ij} (Eq. 12) also grows linearly over time and the numerical instability appears.

2 Reformulation of the 3+1 formalism in the linear regime

As we have discussed, even a small error that inevitably appears at initial or during the numerical calculations linearly grows due to the term as highlighted in red below

$$\partial_t^2 \gamma_{ij} = \Delta\gamma_{ij} - \gamma^{kl}\left(\partial_{lj}\gamma_{ik} + \partial_{li}\gamma_{jk} - \partial_{ij}\gamma_{kl}\right), \tag{21}$$

and eventually crushes the calculation.
To overcome the issue, we have to find a way to ensure the hyperbolicity at least in the linearized system ($\gamma_{ij} = \delta_{ij} + h_{ij}$ with $|h_{ij}| \ll 1$):

$$\partial_t^2 h_{ij} = \Delta h_{ij} - \left(\partial_{kj} h_{ik} + \partial_{ki} h_{jk} - \partial_{ij} h_{kk}\right). \tag{22}$$

To this end, we begin with introducing new auxiliary variables defined by

$$F_i \equiv \partial_j h_{ij} \tag{23}$$
$$h \equiv h_{ii}. \tag{24}$$

We evolve these two variables as independent values. These new auxiliary variables can rewrite the above equation as follows.

$$\partial_t^2 h_{ij} = \Delta h_{ij} - \left(\partial_j F_i + \partial_i F_j - \partial_{ij} h\right). \tag{25}$$

Using Eq. (7), the momentum constraint in the linear system is then expressed as

$$\mathcal{M}_j = D_i K^i_j - D_j K = 0 \implies \dot{h}_{ij,i} - \dot{h}_{ii,j} = \dot{F}_i - \partial_i \dot{h} = 0. \tag{26}$$

We can interpret this equation as the evolution equation for the new auxiliary variable F_i. Furthermore, we take a trace of equation (22), which results in

$$\partial_t^2 h = 2\Delta h - 2\partial_i F_i. \tag{27}$$

Now we apply the Hamiltonian constraint (Eq. 18) and obtain

$$\partial_t^2 h = 2(\Delta h - \partial_i F_i) = 2\mathcal{H} = 0. \tag{28}$$

This indicates that if we can numerically evolve h so that h satisfies

$$\partial_t^2 h = 0 \implies \partial_t h = 0 \implies h = const., \tag{29}$$

we can simultaneously evolve F_i, which obeys

$$\partial_t F_i = 0 \tag{30}$$

from the evolution Eq. (26). Since Eq. (26) is identical to the momentum constraint, numerically satisfying the momentum constraint is also another key here. *[Answer to the question raised during the lecture.]* F_i is initially set to be tiny, or rather to be zero, because

$F_i(t=0) = \partial_i h_{ij} \sim 0$. Then from Eq. (30), those tiny values can be preserved during the evolution. Consequently the terms appearing in parenthesis of Eq. (25) can also be negligible. Thence the original propagation equations (22) (or 25) of gravitational waves can be reformulated as

$$\partial_t^2 h_{ij} = \Delta h_{ij}. \tag{31}$$

The reformulated equation is obviously free from the aforementioned origin of the issue (terms in the parenthesis in Eq. (22)) and acquires the hyperbolicity, i.e., numerically stable.
The essence of above reformulation process can be summarized as follows.

- During the numerical evolution, the Hamiltonian and momentum constraints have to be numerically satisfied.
- On top of that we apply additional algebraic constraint $h = const.$.
- Consequently the extra term, which was the origin of numerical issue, disappears as clearly indicated by Eq. 31.

3 BSSN formalism in the non-linear regime

What we discussed in the previous section can be straightforwardly applied to the non-linear case. Now we proceed to the (*original*) BSSN formalization that is based on the conformal decomposition formulation. We begin with introducing new fundamental variables defined by:

$$\begin{aligned}
\tilde{\gamma}_{ij} &\equiv e^{-4\phi}\gamma_{ij} && (32)\\
\tilde{A}_{ij} &\equiv e^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) && (33)\\
K &\equiv \gamma^{ij}K_{ij} && (34)\\
\phi &\equiv \frac{1}{12}\ln\gamma. && (35)
\end{aligned}$$

After some algebras, the final form of the (original) BSSN equation reads as

$$\begin{aligned}
(\partial_t - \beta^k\partial_k)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} + 2\tilde{\gamma}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k && (36)\\
(\partial_t - \beta^k\partial_k)\phi &= \frac{1}{6}\left(-\alpha K + \partial_k\beta^k\right), && (37)
\end{aligned}$$

for the metric evolution corresponding to Eq. (5), and

$$\begin{aligned}
(\partial_t - \beta^k\partial_k)\tilde{A}_{ij} &= 2\tilde{A}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k \\
&\quad e^{-4\phi}\left[\alpha\left(R_{ij} - \frac{1}{3}e^{4\phi}\tilde{\gamma}_{ij}{}^{(3)}R\right) - \left(D_iD_j\alpha - \frac{1}{3}e^{4\phi}\tilde{\gamma}_{ij}\Delta\alpha\right)\right] \\
&\quad +\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) && (38)\\
(\partial_t - \beta^k\partial_k)K &= \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) - \Delta\alpha, && (39)
\end{aligned}$$

for the extrinsic curvature evolution alternative to Eq. (4). Note that here we consider the vacuum space for simplicity and thus there is no term originated from the energy-momentum tensor (c.f. Eq. 4).

3.1 What makes the BSSN formalism numerically stable?

Although these new four evolution equations (36)-(39) are equivalent to the 3+1 formulation (Eqs. 4 and 5) and sufficient to describe the evolution of all necessary geometrical variables, it still does not guarantee stable numerical evolutions! Now let's see what makes the BSSN scheme numerically stable.
As we confirmed in the previous section, the origin of numerical instability is associated with the non-linear term appearing in the Ricci tensor ${}^{(3)}R_{ij}$. Therefore we have to again carefully look at it. In the conformal decomposition formulation (i.e. $\gamma_{ij} = e^{4\phi}\tilde{\gamma}_{ij}$), ${}^{(3)}R_{ij}$ is now expressed as

$$ {}^{(3)}R_{ij} = R^{\phi}_{ij} + \tilde{R}_{ij}, \tag{40} $$

where

$$ \begin{aligned} R^{\phi}_{ij} &= -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{\gamma}_{ij}\tilde{D}^k\tilde{D}_k\phi + 4\tilde{D}_i\phi\tilde{D}_j\phi - 4\tilde{\gamma}_{ij}\tilde{D}^k\phi\tilde{D}_k\phi & (41)\\ \tilde{R}_{ij} &= \frac{1}{2}\left[-\tilde{\gamma}^{kl}\left(\partial_k\partial_l\tilde{\gamma}_{ij} - \partial_j\partial_l\tilde{\gamma}_{ik} - \partial_i\partial_l\tilde{\gamma}_{jk}\right)\right] + \partial_k\tilde{\gamma}^{kl}\partial_i\partial_j\tilde{\Gamma}_l - \tilde{\Gamma}^l_{jk}\tilde{\Gamma}^k_{il}. & (42) \end{aligned} $$

Again we highlight the non-linear term in red, which may cause the numerical instability. Now we decompose the conformal three metric $\tilde{\gamma}^{ij}$ into $\delta^{ij} + f^{ij}$. With this we can rewrite the Ricci tensor associated with $\tilde{\gamma}_{ij}$ as

$$ \begin{aligned} \tilde{R}_{ij} = & \frac{1}{2}\left[-\Delta\tilde{\gamma}_{ij} + \delta^{kl}\left(\partial_j\partial_l\tilde{\gamma}_{ik} + \partial_i\partial_l\tilde{\gamma}_{jk}\right)\right. \\ & \left. - f^{kl}\left(\partial_j\partial_l\tilde{\gamma}_{ik} - \partial_j\partial_l\tilde{\gamma}_{ik} - \partial_i\partial_l\tilde{\gamma}_{jk}\right)\right] \\ & + \partial_k\tilde{\gamma}^{kl}\partial_i\partial_j\tilde{\Gamma}_l - \tilde{\Gamma}^l_{jk}\tilde{\Gamma}^k_{il}. \end{aligned} \tag{43} $$

Here $\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$ is the conformal contracted Christoffel symbol. On the right hand side of equation, the first and second lines are the linear and non-linear term in $\tilde{\gamma}_{ij}$, respectively. Analogous to the previous discussion in the linear regime (see Eq. 23), we here introduce a new auxiliary variable

$$ F_i \equiv \partial^j\tilde{\gamma}_{ij}. \tag{44} $$

With this we can further rewrite $\tilde{R}_{ij}$ as

$$ \begin{aligned} \tilde{R}_{ij} = & \frac{1}{2}\left[-\Delta\tilde{\gamma}_{ij} + \partial_j F_i + \partial_i F_j\right. \\ & - f^{kl}\left(\partial_j\partial_l\tilde{\gamma}_{ik} - \partial_j\partial_l\tilde{\gamma}_{ik} - \partial_i\partial_l\tilde{\gamma}_{jk}\right) \\ & \left. + \partial_k\tilde{\gamma}^{kl}\partial_i\partial_j\tilde{\Gamma}_l - \tilde{\Gamma}^l_{jk}\tilde{\Gamma}^k_{il}\right]. \end{aligned} \tag{45} $$

We note that F_i is numerically evolved as a independent variable. But how can we derive the appropriate evolution equation for F_i? Similar to Eq. (26), the answer is the momentum constraint

$$\begin{aligned}\mathcal{M}_j &= D_i K^i_j - D_j K \\ &= D_i \tilde{A}^i_j - \frac{2}{3} D_j K = 0. \end{aligned} \tag{46}$$

After multiplying the last line by α (the lapse function) and inserting the left hand side of following equation, which is derived from Eq. 36,

$$2\alpha \tilde{A}_{ij} = -(\partial_t - \beta^k \partial_k)\tilde{\gamma}_{ij} + 2\tilde{\gamma}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k, \tag{47}$$

we finally obtain the evolution equation for F_i:

$$\begin{aligned}(\partial_t - \beta^k \partial_k)F_i &= 2\alpha \left[\partial_j (f^{kj}\tilde{A}_{ik}) - \frac{1}{2}\tilde{A}^{jl}\partial_i \tilde{\gamma}_{jl} + 6\partial_k \phi \tilde{A}^k_i - \frac{2}{3}\partial_i K\right] \\ &\quad + \delta^{jk}\left[-2\partial_k \alpha \tilde{A}_{ij} + \partial_k \beta^l \partial_l \tilde{\gamma}_{ij} + \partial_k \left(2\tilde{\gamma}_{l(i}\partial_{j)}\beta^l - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k\right)\right] \end{aligned} \tag{48}$$

3.1.1 Stability check of F_i

We shortly check whether the evolution equation for the newly introduced auxiliary variable F_i can actually be evolved stably. Analogous to Eq. (12), we decompose the conformal spatial metric $\tilde{\gamma}_{ij}$ and conformal trace-free part of the extrinsic curvature $\tilde{A}_{ij}$ into

$$\tilde{h}_{ij} = A_\gamma \delta_{ij} + \partial_i B_{\gamma_j} + \partial_j B_{\gamma_i} + \partial_{ij} C_\gamma + h^{\mathrm{TT}}_{ij} \tag{49}$$

$$\tilde{A}_{ij} = A_k \delta_{ij} + \partial_i B_{kj} + \partial_j B_{ki} + \partial_{ij} C_k + a^{\mathrm{TT}}_{ij}. \tag{50}$$

Again $A_{\gamma,k}$ and $C_{\gamma,k}$ are scalars, B_{γ,k_i} are divergence-free vectors, and h_{ij} and a_{ij} are the TT tensors. With these, we can write F_i as

$$F_i = \partial^j \tilde{h}_{ij} = \partial_i A_\gamma + \partial_i \Delta C_\gamma + \Delta B_{\gamma_i}. \tag{51}$$

In the previous Sec. 1.3, we have exhibited that the secular evolution of scalar mode C can be the origin of numerical instability. Indeed at this moment, F_i seems to be still suffering from it, indicating F_i may diverge along with the secular growth of C.
However, if we consider its time evolution

$$\dot{F}_i = \partial_i \dot{A}_\gamma + \partial_i \Delta \dot{C}_\gamma + \Delta \dot{B}_{\gamma_i}, \tag{52}$$

the problematic term will be disappeared. Analogous to what we have seen at Eq. (13) together with evolution equations (36) and (38), we obtain several relations between the newly introduced perturbed terms above, some of which are

$$\dot{A}_\gamma = -2A_k \tag{53}$$

$$\dot{C}_\gamma = -2C_k. \tag{54}$$

Furthermore we apply the momentum constraint

$$\mathcal{M} = A_k + \Delta C_k - \frac{2}{3}K = 0. \tag{55}$$

Note that the capital $K(=\gamma^{ij}K_{ij})$ represents the trace of extrinsic curvature. Consequently the evolution of F_i in the linear system can be described by

$$\begin{aligned}\dot{F}_i &= -2\partial_i A_k - 2\partial_i \Delta C_k + \Delta\dot{B}_{\gamma_i} \\ &= -\frac{4}{3}\partial_i K + \Delta\dot{B}_{\gamma_i}.\end{aligned} \tag{56}$$

From this, we can infer that F_i can be evolved without encountering numerical instability as long as the momentum constraint is well satisfied. This is the most crucial part of the (original) BSSN scheme.

3.2 Alternatives

Preserving the essence of original BSSN scheme, there are currently several alternative variables proposed for more stable numerical evolution as well as for a simpler expression of some variables, e.g., Ricci tensor/scalar.

- $\Gamma^i = -\partial_j\tilde{\gamma}^{ij}$ (Baumgarte & Shapiro, 1998) alternative to F_i
- BSSN-puncture formulation: $\chi = e^{-4\phi}$ (Campanelli *et al.* 2006) or $W = e^{-2\phi}$ (Maronetti *et al.*, 2008) alternative to ϕ.