Solution: 3+1 Decomposition of Maxwell Equations

We consider a foliation of spacetime into spacelike hypersurfaces with unit normal

$$n^a = \alpha^{-1}(\partial^a t - \beta^a),$$

where α is the lapse function and β^a the shift vector.

1. Lie derivative of E^a along αn^a

The Lie derivative of a vector field E^a along αn^a is:

$$\mathcal{L}_{\alpha n} E^a = \alpha n^b \nabla_b E^a - E^b \nabla_b (\alpha n^a).$$

Expanding the derivative:

$$\nabla_b(\alpha n^a) = n^a \nabla_b \alpha + \alpha \nabla_b n^a,$$

so:

$$\mathcal{L}_{\alpha n} E^a = \alpha n^b \nabla_b E^a - E^b n^a \nabla_b \alpha - \alpha E^b \nabla_b n^a.$$

2. Maxwell's equations in vacuum

$$\nabla_a F^{ab} = 0,$$
$$\nabla_b \star F^{bd} = 0,$$

where the dual is defined by:

$$\star F^{ab} = \frac{1}{2}\epsilon^{abcd} F_{cd}.$$

We define:

$$E^a = F^{ab} n_b, \quad B^a = - \star F^{ab} n_b.$$

We use the 3+1 decomposition:

$$F^{ab} = n^a E^b - n^b E^a + \epsilon^{abc} B_c,$$
$$\star F^{ab} = n^a B^b - n^b B^a - \epsilon^{abc} E_c,$$

where $\epsilon^{abc} = n_d \epsilon^{dabc}$ is the induced spatial Levi-Civita tensor.

Note that:

$$E^a n_a = B^a n_a = 0.$$

3. Evolution equation for E^a

Projecting $\nabla_a F^{ab} = 0$ orthogonal to n_b, we obtain:

$$\gamma^c{}_b \nabla_a F^{ab} = 0,$$

which gives:

$$\mathcal{L}_{\alpha n} E^i = \epsilon^{ijk} D_j(\alpha B_k) + \alpha K E^i,$$

in vacuum, where D_j is the spatial covariant derivative and K is the trace of the extrinsic curvature.

4. Constraint equation for E^a

Projecting along n_b:

$$n_b \nabla_a F^{ab} = \nabla_a E^a - F^{ab} \nabla_a n_b = 0 \quad \Rightarrow \quad D_a E^a = 0.$$

5. Evolution equation for B^a

We now consider the dual Maxwell equation:

$$\nabla_b \star F^{bd} = 0.$$

Project orthogonal to n_d using $\gamma^c{}_d$:

$$\gamma^c{}_d \nabla_b \star F^{bd} = 0.$$

Substitute the decomposition:

$$\star F^{bd} = n^b B^d - n^d B^b - \epsilon^{bde} E_e.$$

Computing the projection and simplifying yields:

$$\mathcal{L}_{\alpha n} B^i = -\epsilon^{ijk} D_j(\alpha E_k) + \alpha K B^i.$$

6. Constraint equation for B^a

Contract with n_d:

$$n_d \nabla_b \star F^{bd} = \nabla_b B^b - \star F^{bd} \nabla_b n_d = 0,$$

so in vacuum:

$$D_i B^i = 0.$$

7. Summary of 3+1 Maxwell Equations in Vacuum

- **Evolution of E^i:**

$$\mathcal{L}_{\alpha n} E^i = \epsilon^{ijk} D_j(\alpha B_k) + \alpha K E^i$$

- **Evolution of B^i:**

$$\mathcal{L}_{\alpha n} B^i = -\epsilon^{ijk} D_j(\alpha E_k) + \alpha K B^i$$

- **Constraint equations:**

$$D_i E^i = 0, \quad D_i B^i = 0$$

These are the Maxwell equations decomposed in 3+1 form on a foliation of spacelike hypersurfaces, suitable for coupling to general relativity in the ADM formalism.

Solution: *Hamiltonian Constraint*

We are given an initial hypersurface (timeslice) with energy density ρ, momentum density j^i, and extrinsic curvature K_{ij}. We are told:

- The trace of the extrinsic curvature $K \equiv \gamma^{ij} K_{ij}$ is constant.
- The trace-free part of the extrinsic curvature vanishes: $A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij} K = 0$.

1. Simplified Hamiltonian and Momentum Constraints

The full Hamiltonian and momentum constraints in the ADM formalism are:

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \tag{1}$$

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi j^i. \tag{2}$$

Under the assumptions:

$$K_{ij} = \frac{1}{3}\gamma_{ij}K, \quad \text{so} \quad K_{ij}K^{ij} = \frac{1}{3}K^2,$$

since

$$K_{ij}K^{ij} = \frac{1}{9}\gamma_{ij}\gamma^{ij}K^2 = \frac{1}{9}\cdot 3K^2 = \frac{1}{3}K^2.$$

Therefore, the Hamiltonian constraint becomes:

$$R + K^2 - \frac{1}{3}K^2 = 16\pi\rho \quad \Rightarrow \quad R + \frac{2}{3}K^2 = 16\pi\rho. \tag{3}$$

The momentum constraint becomes:

$$D_j\left(\frac{1}{3}\gamma^{ij}K - \gamma^{ij}K\right) = -\frac{2}{3}D^iK = 8\pi j^i. \tag{4}$$

If K is constant, then $D^iK = 0$, so:

$$j^i = 0. \tag{5}$$

2. Degrees of Freedom and Strategy for Solving Constraints

The spatial metric γ_{ij} is a symmetric 3x3 tensor, with 6 independent components. General covariance allows us to choose 3 coordinate conditions (gauge freedoms), reducing this to 3 physical degrees of freedom.

The constraints further reduce the allowable initial data:

- The Hamiltonian constraint imposes 1 condition.
- The momentum constraints impose 3 conditions (but here reduce to trivial constraints since $j^i = 0$ and $K = \text{const}$).

In this special case, only the Hamiltonian constraint remains nontrivial.

Strategy: A common approach is the **conformal method**:

- Choose a conformal spatial metric $\tilde{\gamma}_{ij}$.
- Assume the physical metric is $\gamma_{ij} = \psi^4\tilde{\gamma}_{ij}$, where ψ is the conformal factor.
- Substitute into the Hamiltonian constraint to obtain an elliptic equation for ψ.
- Solve for ψ under desired boundary conditions (e.g., periodic).

3. Lower Bound on $|K|$ with Periodic Boundary Conditions

We assume the spatial manifold is a 3-torus (periodic cube), so the boundary terms vanish under integration by parts.

Integrate the Hamiltonian constraint over the spatial volume:

$$\int (R + \tfrac{2}{3}K^2)\sqrt{\gamma}\, dV = \int 16\pi\rho\sqrt{\gamma}\, dV. \tag{6}$$

Let $V = \int \sqrt{\gamma}\, dV$, the total volume. Then:

$$\int R\sqrt{\gamma}\, dV + \tfrac{2}{3}K^2 V = 16\pi \int \rho\sqrt{\gamma}\, dV. \tag{7}$$

Using the assumption:

$$\int R\sqrt{\gamma}\, dV \leq 0, \quad \rho \geq 0,$$

we get:

$$\tfrac{2}{3}K^2 V \leq 16\pi \int \rho\sqrt{\gamma}\, dV. \tag{8}$$

Thus, in vacuum $\rho = 0$, we find:

$$\tfrac{2}{3}K^2 V \leq 0 \quad \Rightarrow \quad K^2 \leq 0 \quad \Rightarrow \quad K = 0.$$

But the Hamiltonian constraint becomes:

$$R = 0.$$

Yet this contradicts the assumption $\int R\sqrt{\gamma}\, dV \leq 0$, unless $R = 0$ everywhere.
If $R < 0$ somewhere, then we must have $K \neq 0$, and so:

$$K^2 \geq \frac{3}{2V}\left(16\pi \int \rho\sqrt{\gamma}\, dV - \int R\sqrt{\gamma}\, dV\right). \tag{9}$$

So, under the condition $\int R\sqrt{\gamma}\, dV \leq 0$, and assuming $\rho = 0$, the only possibility is:

$$K^2 \geq \frac{3}{2V}\left(-\int R\sqrt{\gamma}\, dV\right) > 0.$$

Conclusion: For such a spacetime with non-zero integrated negative curvature and vanishing matter, the slice must have nonzero mean curvature K. Therefore, the spacetime cannot be stationary (i.e., with $K = 0$), because stationarity would require a time-symmetric slice with $K = 0$, which contradicts the integrated constraint.

Solution: Harmonic Formulation of GR

We are given the harmonic formulation of the Einstein equations (in vacuum) with the modified Einstein tensor:

$$R_{ab} - \frac{1}{2}Rg_{ab} - \frac{1}{2}\left(\nabla_a\Gamma_b + \nabla_b\Gamma_a\right) + \frac{1}{2}g_{ab}g^{cd}\nabla_c\Gamma_d = 0,$$

and the harmonic gauge condition:

$$\Gamma_a := g_{ab}g^{cd}\Gamma^b_{cd} = 0.$$

1. Freedom to Choose $\partial_t g_{ta}$ to Enforce Harmonic Gauge

We are given initial spatial metric components $g_{ij}(t=0)$ and their time derivatives $\partial_t g_{ij}(t=0)$, but not the time components $\partial_t g_{ta}$. The harmonic condition can be used to determine them.

Recall that:

$$\Gamma^a = g^{bc}\Gamma^a_{bc} = -\frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}g^{ab}).$$

So,

$$\Gamma_a = g_{ab}\Gamma^b = -g_{ab}\frac{1}{\sqrt{-g}}\partial_c(\sqrt{-g}g^{bc}).$$

This expression depends on first derivatives of g^{ab}, and therefore on first derivatives of g_{ab}.

Since the harmonic condition $\Gamma_a = 0$ is first order in derivatives of the metric, and since g_{ij} and $\partial_t g_{ij}$ are already fixed by the initial data, the remaining freedom lies in choosing $\partial_t g_{ta}$ to satisfy $\Gamma_a = 0$ at $t = 0$.

Conclusion: Yes, the 4 components $\partial_t g_{ta}$ can always be freely chosen (locally) to enforce the harmonic condition $\Gamma_a = 0$ on the initial slice.

2. Do Choices of $\partial_t g_{ta}$ Affect the Constraints?

The Hamiltonian and momentum constraints arise from the 4 Einstein equations $G^{0\mu} = 8\pi T^{0\mu}$ when using a 3+1 decomposition. These constraints depend on the metric g_{ij}, its time derivative $\partial_t g_{ij}$, and the lapse and shift (or equivalently $g_{0\mu}$).

However, **the constraints are already independent of the second time derivatives of g_{ta}.** The freedom to set $\partial_t g_{ta}$ to satisfy $\Gamma_a = 0$ simply fixes the coordinate (gauge) degrees of freedom and does not directly affect the intrinsic or extrinsic geometry of the initial slice encoded in the constraints.

Conclusion: No, the choice of $\partial_t g_{ta}$ used to enforce the harmonic gauge does not interfere with whether the Hamiltonian and momentum constraints are satisfied.

3. Evolution of Γ_a

Let us define $\Gamma_a := g_{ab} g^{cd} \Gamma^b_{cd}$. Under the harmonic Einstein equations, we can derive an evolution equation for Γ_a.

Key idea:

Under the harmonic formulation, the Einstein equations reduce to a system of quasilinear wave equations for the metric components. The constraint $\Gamma_a = 0$ is analogous to a gauge condition and must be preserved during evolution.

Let's define the modified Einstein tensor as:

$$\mathcal{E}_{ab} := R_{ab} - \nabla_{(a}\Gamma_{b)} + \frac{1}{2} g_{ab} \nabla_c \Gamma^c.$$

Taking the divergence of the Einstein tensor and using the Bianchi identity:

$$\nabla^a G_{ab} = 0 \quad \Rightarrow \quad \nabla^a \mathcal{E}_{ab} = -\nabla^a \nabla_{(a}\Gamma_{b)} + \frac{1}{2} \nabla_b \nabla_a \Gamma^a = -\frac{1}{2} \Box \Gamma_b + (\text{curvature terms}).$$

This leads to the evolution equation for Γ_a:

$$\Box \Gamma_a + R_a{}^b \Gamma_b = 0. \tag{1}$$

Implication

This is a wave equation for Γ_a. If we set:

$$\Gamma_a(t=0) = 0, \quad \partial_t \Gamma_a(t=0) = 0,$$

then by uniqueness of solutions to hyperbolic PDEs, the solution remains zero:

$$\Gamma_a(t) = 0 \quad \forall t,$$

i.e., the harmonic condition is preserved under evolution.

But this requires that the initial data satisfy both the harmonic gauge condition *and* the Hamiltonian and momentum constraints — otherwise the wave equation above may have non-zero source terms and evolve away from zero.

Conclusion: If the harmonic constraint $\Gamma_a = 0$ and its time derivative vanish initially, and the Einstein constraints are satisfied, then $\Gamma_a = 0$ is preserved during evolution under the harmonic Einstein equations.