Homework #2 : Strong Gravity

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Problem # 1

(a) The FRW metric:

$$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2).$$

$$\Box x^b = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, \partial^\mu x^b)$$

$$= \frac{1}{a(t)^3} \partial_\mu (a(t)^3 \partial^\mu x^b)$$

$$= \frac{1}{a(t)^3} [3 a(t)^2 \partial_0 a\, \partial^0 x^b + a(t)^3 \partial_i \partial^i x^b]$$

$$= \frac{3}{a} \frac{\partial a}{\partial t} \left(- \frac{\partial x^b}{\partial t} \right) + \underbrace{g^{ij} \partial_i \partial_j x^b}_{=0}$$

$$\boxed{\Box x^b = -\frac{3}{a} \frac{\partial a}{\partial t} \delta^b_0}$$

Let, $ds^2 = -\left(\frac{dt}{d\tau}\right)^2 d\tau^2 + a^2(t(\tau)) [dx^2 + dy^2 + dz^2]$

$$\therefore \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, \partial^\mu x^c) = 0 \quad \text{[Harmonic condition]}$$

$$\Rightarrow \frac{1}{\frac{\partial t}{\partial \tau} a^3} \frac{\partial}{\partial \tau} \left(\frac{\partial t}{\partial \tau} a^3 \cdot \left(\frac{d\tau}{dt}\right)^2 \right) = 0 \quad \text{[partial} \rightarrow \text{total derivative]}$$

$$\Rightarrow \frac{1}{a^3 \frac{\partial t}{\partial \tau}} \frac{\partial}{\partial \tau} \left(a^3 \frac{\partial \tau}{\partial t} \right) = 0$$

$$\Rightarrow \frac{1}{a^3} \frac{d\tau}{dt} \cdot \frac{d}{d\tau} \left(a^3 \frac{d\tau}{dt} \right) = 0$$

$$\Rightarrow \frac{1}{a^3} \frac{d}{dt} \left(a^3 \frac{d\tau}{dt} \right) = 0$$

$$\Rightarrow \boxed{\frac{d^2\tau}{dt^2} + \frac{3}{a} \frac{da}{dt} \cdot \frac{d\tau}{dt} = 0}$$

Let, $\frac{da}{dt}\Big/ a = \pm \frac{\sqrt{\Lambda}}{\sqrt{3}}$

$$\frac{d^2\tau}{dt^2} \pm \sqrt{3\Lambda}\,\frac{d\tau}{dt} = 0 \Rightarrow \frac{d}{dt}\left(\frac{d\tau}{dt} \pm \sqrt{3\Lambda}\,\tau\right) = 0$$

$$\Rightarrow \frac{d\tau}{dt} \pm \sqrt{3\Lambda}\,\tau = C$$

$$\Rightarrow \frac{d(C \mp \sqrt{3\Lambda}\tau)}{C \mp \sqrt{3\Lambda}\,\tau} = \mp\sqrt{3\Lambda}\,dt$$

$$\Rightarrow \ln\left(C \pm \sqrt{3\Lambda}\,\tau\right) = \mp\sqrt{3\Lambda}\,t + D$$

$$\Rightarrow \boxed{\tau = C' + D' e^{\mp\sqrt{3\Lambda}\,t}}$$

C', D' are some constant depend on initial conditions.

(b) $$ds^2 = -\frac{r^2 - 2Mr}{r^2}\,dt^2 + \frac{\sin^2\theta}{r^2}(r^2 d\phi)^2 + \frac{r^2}{r^2 - 2Mr}\,dr^2 + r^2 d\theta^2$$

$$dt = dt_{BL} + \frac{4M^2}{r^2}\,\frac{dr}{1 - \frac{2M}{r}}$$

$$dx = dr\cos\phi\sin\theta + (r - M)\cos\theta\cos\phi\, d\theta - (r-M)\sin\phi\sin\theta\, d\phi$$

$$dy = dr\sin\phi\sin\theta + (r - M)\cos\theta\sin\phi\, d\theta + (r-M)\cos\phi\sin\theta\, d\phi$$

$$dz = dr\cos\theta - (r - M)\sin\theta\, d\theta$$

$$dx^2 + dy^2 + dz^2 = dr^2 + (r-M)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

We observe that, $\boxed{x^2 + y^2 + z^2 = (r - M)^2 := R^2}$.

$$\Rightarrow (d\theta^2 + \sin^2\theta\, d\phi^2) = \frac{1}{(r-M)^2}(dx^2 + dy^2 + dz^2 - dr^2)$$

Now,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt_{BL}^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + \frac{r^2}{(r-M)^2}(dx^2 + dy^2 + dz^2 - dr^2)$$

$$= -\left(1-\frac{2M}{\rho}\right)dt_{BL}^2 + \left(\frac{1}{1-\frac{2M}{\rho}} - \frac{1}{1-\frac{2M}{\rho}+\frac{M^2}{\rho^2}}\right)d\rho^2$$

$$+ \frac{(R+M)^2}{R^2}\left(dx^2+dy^2+dz^2\right)$$

$$\left.\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned}\right\} \quad dx^2+dy^2+dz^2 = dR^2 + R^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

$$= -\left(1-\frac{2M}{\rho}\right)dt_{BL}^2 + \frac{M^2/\rho^2}{\left(1-\frac{2M}{\rho}\right)\left(1-\frac{M}{\rho}\right)^2}d\rho^2 + (R+M)^2 dR^2/R^2 +$$

$$(R+M)^2 d\Omega_2^2$$

$$= -\left(\frac{\rho-2M}{\rho}\right)dt_{BL}^2 + \left[\frac{M^2\rho}{(\rho-2M)(\rho-M)^2} + \frac{(R+M)^2}{R^2}\right]dR^2$$

$$+ (R+M)^2 d\Omega^2$$

$$= -\left(\frac{R-M}{R+M}\right)dt_{BL}^2 + \left[\frac{M^2(R+M)}{R^2(R-M)} + \frac{(R+M)^2}{R^2}\right]dR^2 + (R+M)^2 d\Omega^2$$

$$= -\frac{R-M}{R+M}dt_{BL}^2 + \frac{R+M}{R^2}\left[\frac{M^2+R^2-M^2}{R-M}\right]dR^2 + (R+M)^2 d\Omega^2$$

$$= -\frac{R-M}{R+M}dt_{BL}^2 + \frac{R+M}{R-M}dR^2 + (R+M)^2 d\Omega^2$$

$$= -\frac{R-M}{R+M}\left(dt - \frac{4M^2}{(R+M)}\frac{dR}{R-M}\right)^2 + \frac{R+M}{R-M}dR^2 + (R+M)^2 d\Omega^2$$

$$\boxed{ds^2 = -\frac{R-M}{R+M}dt^2 + \frac{8M^2\,dt\,dR}{(R+M)^2} - \frac{(4M^2)^2}{(R+M)^3(R-M)}dR^2 + \frac{R+M}{R-M}dR^2 + (R+M)^2 d\Omega^2}$$

$$\sqrt{-g} = (R+M)^2\sin\theta\sqrt{\left(\frac{(4M^2)^2}{(R+M)^2} + 1 - \frac{(4M^2)^2}{(R+M)^6}\right)}$$

$$\sqrt{-g} = (R+M)^2 \sin\theta$$

Now,

$$\Box t = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, g^{\mu t})$$

$$= \frac{1}{\sqrt{-g}} \partial_R (\sqrt{-g}\, g^{Rt})$$

$$= \frac{1}{(R+M)^2 \sin\theta} \partial_R \left((R+M)^2 \sin\theta \times \frac{(-4M^2)}{(R+M)^2} \right)$$

$$= 0$$

$$\Box x = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \left(g^{\mu R} \partial_R x + g^{\mu\phi} \partial_\phi x + g^{\mu\theta} \partial_\theta x\right)\right)$$

$$= \frac{1}{\sqrt{-g}} \Big(\partial_R (\sqrt{-g}\, g^{RR} \sin\theta \cos\phi) + \partial_\phi (\sqrt{-g}\, \underbrace{g^{\phi\phi}}_{\frac{1}{R^2 \sin^2\theta}} R(-\sin\phi)\sin\theta)$$

$$+ \partial_\theta (\sqrt{-g}\, g^{\theta\theta} R (\cos\theta) \sin\phi) \Big)$$

$$= \frac{1}{\sqrt{-g}} (\sin\theta \cos\phi \, 2R - R\cos\phi \sin\theta - R \cos\phi \sin\theta)$$

$$= 0$$

$$\boxed{\Box y = 0 \quad \text{and} \quad \Box z = 0}$$ [Similarly]

Hence it is a Harmonic co-ordinate.

BH horizon is at $\boxed{\sqrt{x^2+y^2+z^2} = R = M}$ $(r = 2M)$

Now, we choose, $ds^2 = 0$ and $\theta = \phi = 0$. Then,

$$0 = -\frac{R-M}{R+M} \left(\frac{dt}{dR}\right)^2 + \frac{8M^2 \left(\frac{dt}{dR}\right)}{(R+M)^2} - \frac{(4M^2)^2}{(R+M)^3 (R-M)} + \frac{R+M}{R-M}$$

$$\Rightarrow \left(\frac{dt}{dR}\right)^2 - \frac{8M^2}{R^2 - M^2} \frac{dt}{dR} + \frac{(4M^2)^2}{(R^2-M^2)^2} - \frac{(R+M)^2}{(R-M)^2} = 0$$

$$\Rightarrow \left(\frac{dt}{dR} - \frac{4M^2}{R^2-M^2}\right)^2 - \frac{(R+M)^2}{(R-M)^2} = 0$$

$$\Rightarrow \left(\frac{dt}{dR} - \frac{4M^2}{R^2-M^2} + \frac{R+M}{R-M}\right)\left(\frac{dt}{dR} - \frac{4M^2}{R^2-M^2} - \frac{R+M}{R-M}\right) = 0$$

$$\Rightarrow \left(\frac{dt}{dR} + \frac{(R+M)^2 - 4M^2}{R^2-M^2}\right)\left(\frac{dt}{dt} - \frac{(R+M)^2+4M^2}{R^2-M^2}\right) = 0$$

$$\Rightarrow \frac{dt}{dR} = -\frac{R+3M}{R+M} \quad \text{or} \quad \frac{(r+M)^2+(2M)^2}{R^2-M^2}$$

We can see that. $\left.\frac{dt}{dR}\right|_{r=M} = -\left.\frac{R+3M}{R+M}\right|_{R=M} = -2$ is finite.

The null is not totally shrunk, so the coordinate could penetrate the horizon. Another argument:

At horizon, $\nabla_{BL} t = \partial_{t_{BL}} t = 1$

$$\nabla_\rho t = \partial_\rho t = \left(2M \times \frac{\rho}{\rho-2M} \cdot \left(-\frac{2M}{\rho^2}\right)\right) = \left(-\frac{4M^2}{\rho^2}\frac{1}{1-\frac{2M}{\rho}}\right)$$

$$n^a = N\nabla^a t = N g^{ab}\partial_a t$$

$$n^{t_{BL}} = N\left(\frac{2M}{\rho}-1\right)^{-1}$$

$$n^\rho = N\left(1-\frac{2M}{\rho}\right)\left(-\frac{4M^2}{\rho^2}\right)\frac{1}{\left(1-\frac{2M}{\rho}\right)} = -\frac{4M^2}{\rho^2}N$$

$$\frac{1}{N} = n^{t_{BL}}\nabla_{BL} t + n^\rho \nabla_\rho t$$

$$= N\left(\frac{2M}{\rho}-1\right)^{-1} + \left(-\frac{4M^2}{\rho^2}\right)N\left(-\frac{4M^2}{\rho^2}\right)\left(1-\frac{2M}{\rho}\right)^{-1}$$

$$\Rightarrow N^2 = \frac{\left(1-{}^{2M}/_{\rho}\right)}{1-\left(\frac{4M^2}{\rho^2}\right)^2} = \frac{1}{\left(1+\frac{2M}{\rho}\right)\left(1+\left(\frac{2M}{\rho}\right)^2\right)}$$

Therefore, the lapse function.

$$N = \frac{1}{\sqrt{\left(1+\frac{2M}{r}\right)\left(1+\left(\frac{2M}{r}\right)^2\right)}}$$

N is finite at horizon. Hence, if we use t as coordinate for the metric, the coordinate wouldn't misbehave because two constant time slice would have orthonormal time like vector that goes from one slice to another penetrating the horizon bound. Hence, this t coordinate can act as a propertime for some observer.

> Therefore, the Laps function doesn't blow up. Hence, the co-ordinate could penetrate the horizon.

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